The Treble with Topology:
An Exploration of the Preservation of Harmonic Structure through Topological Mapping

A Senior Comprehensive Project
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I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

name

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Abstract

The interrelationship between music and mathematics is not a new field of study. From the ratios between octaves (2:1), fifths (3:2), and fourths (4:3) to the physics of consonance and dissonance as seen on a spectrograph, one would not be hesitant to conclude that these two subjects are connected on a scientific level. However, overall there has been less inquiry into the study of the artistic connection of math and music. This paper will explore that connection through the topological modeling of two pieces of music from the Classical and Romantic periods using techniques pioneered by Dimitri Tymoczko. These models will then be used to re-transcribe two new pieces—one from each mapping—and using the Classical and Romantic style periods as a guide, this paper will then conclude whether or not mathematics can be linked to the artistic level of music as well.

Contents

1	Introduction	1
2	Musical Analysis of *Ave Verum Corpus* W. A. Mozart (1756 – 1791)	5
3	Musical Analysis of *Ave Verum* Franz Liszt (1811 – 1886)	15
4	Overview of Tymoczko’s methods in “A Geometry of Music”	23
5	Introduction to Orbifolds: Intro to Topology	28
6	Introduction to Orbifolds: Where Chords Reside	36
7	Analysis and Transcription of Mozart’s *Ave Verum Corpus*	43
8	Analysis and Transcription of Liszt’s *Ave Verum*	49
9	Musical Analysis of Classical Period Musical Composition	56
10	Musical Analysis of Romantic Period Musical Composition	62
11	Conclusion	68
Scores		70
References		84

1 Introduction

With accepted connections between music and math regarding the physics of sound and the composition of atonal music, it is only natural to wonder next whether there is a connection between the creativity involved in composing tonal music and mathematics. This paper explores the possible connection between topological mapping and tonal composition to conclude whether or not the art of music could be influenced by unseen mathematical characteristics.

In order to rightfully come to a conclusion, two pieces of differing tonal styles will be analyzed, mathematically modeled, and then re-transcribed to see whether or not any of their musical characteristics hold. These two style periods, chosen for their contrast in theory and aural experience yet their similarity in time period and adherence to rules, are Classical and Romantic.

Classical music is often the most accessible and practiced music for musicians. From Beethoven to Bach, the pieces created by these composers have stood the test of time. The Classical piece in this paper is a motet by Wolfgang Amadeus Mozart (1756-1791), a composer whose name is familiar even to non-musicians. Born to father Leopold, who was a respected composer and violinist, and who was employed by the Archbishop of Salzburg, Mozart first discovered music at the age of three through his father's piano lessons to his sister. He is said to have been a composer from the start, however his playing surpassed that quickly. Mozart gave his first public performance at the age of five at the University of Salzburg.

This successful performance sparked a "series of lengthy and grueling tours" featuring Mozart and his sister, orchestrated by Mozart's father (Siepmann, p. 6). It was through these tours that Mozart made a name for himself, or rather, his father did. All of Leopold's attention was on young Mozart, and although Mozart is said to have never complained, he would eventually be plagued with anxiety until his death. He went on to have a relationship with his father filled with extremes, that is, one with love, hatred, and fear. Furthermore, Mozart spent most of his childhood around only adults, which prompted him to make a dreamland consisting of only children. This inclination toward youth dominated his personality, and "his sense of fun was a major part of his personality from infancy until death", so much so that he has been labeled an "eternal child" (Siepmann, pp. 11,44).

Mozart's prodigy did not end at the piano however, for at the age of seven he seemingly picked up a violin and began playing with no instruction. This discovery prompted another series of tours, again with Mozart at the center. Brilliant beyond his years, Mozart had composed in virtually every genre before even reaching his teens. In 1773, Mozart began to feel his first setback, when he started to look for employment positions and couldn't find any. That year however, he matched his failure with extreme compositional output, unlike his father who wallowed in self-pity while he resumed a post under the Archbishop of Salzburg. Mozart was ready to get away from his father, but had to wait for an opportunity to do so. Eventually, it would be Leopold's own slip up that caused his separation from his son.

For the first time free of his father's unflinching grasp, Mozart now looked to his mother for support in looking for new employment. However through letters Leopold continued to be a key component of Mozart's life and decisions. It was because of this that Mozart began a rebellion of sorts, refusing lucrative positions and not listening to his father's urgent suggestions. He gained a sense of detachment; he was able to compose while his wife was in labor in an adjoining room. His father's emotional abuse would sink him into states of depression, one of the worst of which was after the death of his mother, when he lost his only comforting parental figure.

Years later, when his father died, his only response was a "business-like" letter to his sister, which truly shows how far they eventually grew apart (Siepmann, p. 126). However Mozart's work went on successfully, for he stayed true to his prodigal name despite times of low finances and illness. In the end, after falling ill in September of 1791, Mozart continued mentally composing his final piece (*Requiem*, *K. 626)* until the day of his death three months later.

A "practical composer", Mozart is a clear choice to exemplify the Classical style (Siepmann, p. 55). His playful pieces reflect his playful personality, yet still he adheres to the strict conventions present in Classical music, as he did with the rules set by his father throughout his childhood. It is for these reasons that this piece was chosen for analysis in this paper.

Historically, periods of great simplicity are followed by periods of complexity, as the pendulum tends to swing back and forth. Romanticism was the pendulum's swing away from the simplicity of Classicalism. Romanticism focused on the communication of emotion first and foremost, and introduced more dissonances with less predictability to their placement. The Romantic piece analyzed in this paper is again a motet by Franz Liszt (1811-1886), who is less well known than Mozart, but still an important figure in music history. Both pieces chosen were four-voice, mostly homorhythmic motets so as to limit the differences between the two pieces that weren't subjective to the composer and the style of composition.

Liszt was born in Austria but spent a lot of time in France, Germany, and Italy throughout his life and career. Another child prodigy at the piano, he later became additionally renowned as a composer, teacher, and writer. Taught first by his father, his formal education took a back seat to his music education after his genius was recognized at age six. His first public appearance in a concert was in Oedenburg at the age of nine, where Liszt played a Concerto by Ries and drew upon the audience for suggestions for themes on which he would then improvise. His second teacher was Carl Czerny, a student of Beethoven; in order to study with Czerny, Liszt had to travel to Vienna. Czerny, however, never charged Liszt for lessons, and the two later became friends. Liszt received similar cost-free instruction in composition from Antonio Salieri, who had also taught Beethoven and Schubert. One example of his recognized genius is that at the age of 11 Liszt had a variation of a Beethoven waltz published; upon hearing it, Beethoven kissed Liszt on the forehead.

Liszt grew up in a strict Catholic household, and upon his widespread recognition as a child prodigy, his family followed wherever his music would take them. His father provided the business acumen needed to execute such a lifestyle, and it wasn't until his death that Liszt realized that as a musician, no matter how renowned, he was of lower class. In order to make ends meet, Liszt took up teaching, which is how he met his first real love Caroline de Saint-Cricq. His love for her stood the test of time, and in his will he left her a ring. After their parting Liszt fell into a state of deep depression.

Liszt continued to teach, compose, and perform for many years until he found his next love, Countess Marie d'Agoult in 1833. During his affair with her he had three kids, however and they separated in 1844. It was during this period that Liszt is said to have written some of his most brilliant music, a clear contrast to his former parting with a love. He toured around Europe and found great acclaim and success through his performances. Liszt had one more true love, with whom he never wed for religious reasons, a Princess of Polish descent, Carolyne zu Sayn-Wittgenstein. She encouraged him to end his performance career and instead focus on composition.

Liszt's life aside from his sadness in loss of love later took a turn for the worst when he lost a son and a daughter in 1859 and 1862, respectively. He resolved to live in solitude, however that did not stop his composition or his travels. The last phase of his life he alternated between Germany, Italy, and Hungary, where he continued to teach, compose, and attend concerts of his work. While he was physically healthy, his loneliness accompanied by his "nomadic life" left Liszt in a state of depression, however it is said that his pupils sustained him. In July 1886, he came down with a hacking cough, and nine days later he died.

It's clear from Liszt's life that he was both a brilliant and depressive individual, which gives greater perspective to the music he composed. Furthermore, his own life leant inspiration to the Romantic period, which his pieces exemplify. It is for these reasons that a piece by Liszt was chosen for this paper.

While this paper is focused on the relationship of mathematics and music, it is important to note that there are connections between math and music that are already firmly acknowledged, many of which are related to the music of Arnold Schoenberg (1874-1951). Schoenberg is credited with "the emancipation of dissonance", which he did through the use of a 12-tone row. Mathematically, these rows hold much significance, especially since the discovery of the Hexachordal Theorem, which states that both hexachords in any row will have the same number of each interval between their pitches. Additionally one can also explore Hanson chemistry and set theory as well. However this is not what will be explored in this paper: instead of looking at pitches in rows or as integers, we will be utilizing traditional music theory along with topological mapping.

This paper thus utilizes the work of Dimitri Tymoczko on the topological mapping of music to analyze music mathematically and later re-transcribe two new pieces. Tymoczko, a professor of composition and music theory at Princeton University, credits four individuals with his evolution from rock music to classical music with a mathematical undertone, one of whom is recognizable as a prominent atonal composer: Milton Babbitt. Studying music and philosophy at Harvard University, Tymoczko received his Ph.D in music composition from the University of California, Berkeley. A composer as well as a professor, Tymoczko's book "A Topology of Music" is recognized as a monumental milestone in the connection between mathematics and music theory. For that reason, this work will be our primary source for topological modeling of music, however the methods used to compose two new pieces will be entirely original.

Challenging the comfort of musicians and composers, this paper seeks to prove that music and math are not only both sciences, but that they are both arts. Furthermore, in attempting to compose pieces of music using topological maps, this paper will seek to prove that it is at least possible to do so, and in doing so potentially even open a door for a new genre of music in the future, perhaps to be called topotonal music. Similar to the creativity involved in composing a piece, there is creativity involved in choosing how to alter the topological map of a piece, both of which are subjective to the composer or mathematician. This fact makes the idea of topotonal music more intriguing as a genre, so that even if the results of this paper are not pleasing to the ear, in determining whether topological mapping has the ability to preserve the harmonic structure this will determine its future as a genre in music composition.

2 Musical Analysis of *Ave Verum Corpus* W. A. Mozart (1756-1791)

In order to confidently determine whether topological mapping has the ability to preserve the harmonic structure of a Classical motet, a piece that exemplifies such harmonic structure must be used as the starting point. For that purpose, we will be using Wolfgang Amadeus Mozart's *Ave Verum Corpus K. 618*, a 4-voice choral motet in the Classical style.

A Roman numeral analysis of this piece is provided on p. 70. A Roman numeral analysis is a way of representing keys, chords, and non-chord tones used throughout a piece by placing corresponding Roman numerals beneath each chord and identifying the embellishing tones. In other words, it is a visual harmonic road map of a piece. The Roman numeral used indicates on which scale degree the chord has its root (for instance a D chord in the key of D major, also called the tonic chord, would be identified as I and the chord of A, or the dominant, would be identified as V). Capital Roman numerals signify major chords while lower-case Roman numerals signify minor chords. Diminished chords are followed by "^{o}" and augmented chords are followed by "$^{+}$".

Furthermore, next to the Roman numerals are numbers indicating what position the chord is in, that is, which note is on the bottom. For a three-note chord, if the chord has the root in the bottom, there are no numbers written (however there is an implied $^{5}_{3}$), if the third is in the bottom there is a "^{6}", and if the fifth is in the bottom there is a "$^{6}_{4}$". Furthermore, for a seventh or four-note chord, root position is signified similarly while if the third is in the bottom there is a "$^{6}_{5}$", if the fifth is in the bottom there is a "$^{4}_{3}$", and if the seventh is in the bottom there is a "$^{4}_{2}$". If the chord is from a different key, that key will be designated as a Roman numeral under a line under the Roman numeral of the chord in that key. For example, in D major, V/V is the dominant chord in A major—in other words a E major chord. Finally, modulations are signified by the designation of a new key entirely beneath the original line of Roman numeral analysis. These often include pivot chords where the chord occurs in both keys but serves as a transition from one to the other, which are signified by a box around the two Roman numerals, the one from the original key over the other in the new key.

Now, using this information, we analyze *Ave Verum Corpus* by identifying the defining characteristics of a motet and how this piece holds those characteristics. A motet is a "sacred polyphonic composition with Latin text, which may or may not have *colla voce* or independent instrumental accompaniment" (Sanders).

In this case, the text is:

"Ave verum corpus natum de Maria Virgine, vere passum immolatum in cruce pro homine, Cujus latus perforatum unda fluxit et sanguine, Esto nobis praegustatum in mortis examine."	"Hail true body born of the Virgin Mary, Who has truly suffered, was sacrificed on the cross for mortals, Whose side was pierced, Whence flowed water and blood, Is for us a foretaste (of heaven) During our final examining."

Translation: Ron Jeffers

Mozart's *Ave Verum Corpus* is mostly homophonic and in four voices: Soprano, Alto, Tenor, and Bass. It begins in D major, has a period of modulation ending in d minor, followed by a return to the original key. This structure is a convention of Classical motets, and it allows for the tonic home of D to be clear as the "home key" throughout the piece. This tonic stability is a staple of Classical music. Additionally, the modulation is approached clearly rather than deceptively, as can be seen at the beginning of the third phrase, through the use of stepwise motion in m. 18: bb. 1-3 (see Figure 2-4, p. 9).

The first eight measures of the piece are in the home key and contain an imperfect authentic cadence in mm. 3-4: bb. 4-4 and a half cadence in mm. 7-8: bb. 3-4. Then, in m. 9 Mozart begins the next phrase in A major, the dominant key, thus beginning the modulatory section of the piece. While in A major, we have a deceptive cadence in m. 12: bb. 2-4 and a perfect authentic cadence in mm. 15-16: bb. 3-4. We remain in A major until the C# then moves to a C♮ in the Tenor, becoming a V^4_3 chord in the newly established key of F major—the submediant of our previous key (m. 18, bb. 3-4). Then, following two inconclusive cadences: an imperfect authentic cadence (IAC) in mm. 19-20: bb. 4-4 and a half cadence in mm. 23-24: bb. 4-2, the modulatory section ends. We then return to the home key of D major in m. 25 (see Figure 2-1 below).

There are two types of authentic cadence that have been mentioned here and that will be seen throughout the piece. A perfect authentic cadence (PAC) has the progression V-I where both the Soprano and Bass voices land on the tonic. Alternatively, in an imperfect authentic cadence (IAC), the progression remains V-I, however either the Soprano or the Bass voice will be on a pitch other than the tonic. This will be key in our analysis of this piece (Clendinning, p. 246).

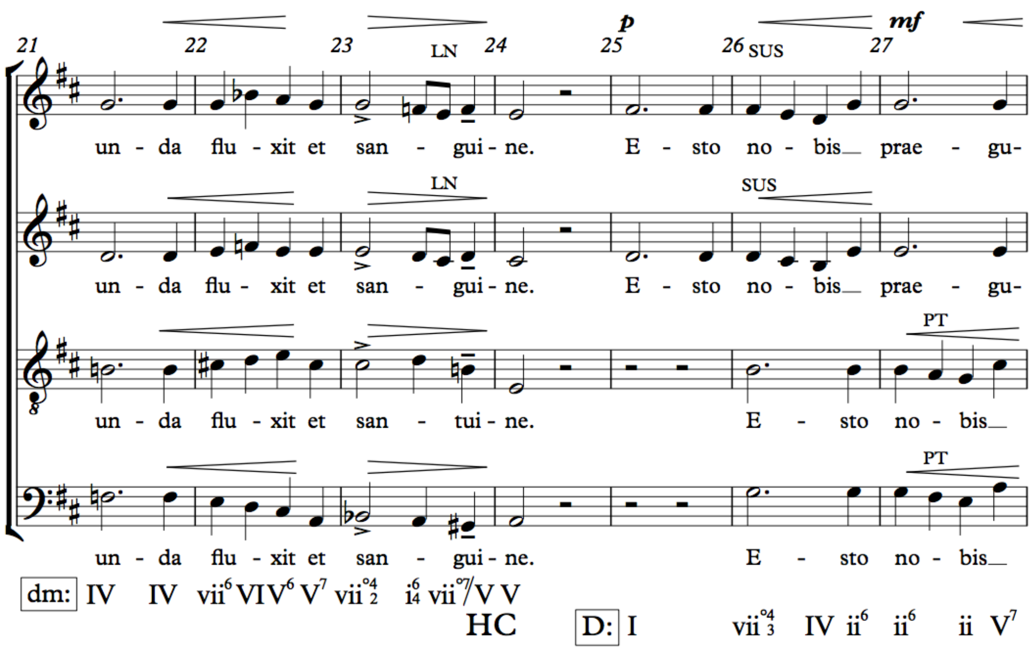


Figure 2-1

In the Classical period the harmonic fabric of the piece focuses more on the counterpoint of the piece and less on the meaning of the words accompanying the composed melodies. The structure of traditional motets varies from period to period, but in the Classical period "every section of text introduced a new motif and it lead through the parts in contrapuntal writing" (Leichtentritt, p. 183). This is best recognized when listening to the piece, for there is no reoccurring melody.

Each part has a distinct role: the Bass functions as the "foundation of the harmonic progressions" and the Soprano contains the melody (Clendining, p. 213). This is obvious in the Bass due to the fact that most of the notes are of long duration, and the only time that there are embellishing tones in the Bass is when the Bass is in parallel motion with the Tenor part as in mm. 27 and 29 (see Figure 2-1 above and Figure 2-4, p. 9).

Additionally, by listening to this piece one can tell that the Soprano part stands out as the melody simply due to its tessitura and the fact that this part is the one that would get stuck in one's head. As a whole, in mm. 2-4, 7, 10-16, 18-20, 25-38, through the use of non-chord tones, shorter note durations, and advanced entrances, the Soprano line stands out melodically and harmonically and thus solidifies its role as the melody.

Another important stylistic staple of Classical period music is the harmonic structure at cadence points. The first cadence is an imperfect authentic cadence in mm. 3-4: bb. 3-4 (see Figure 2-2 below). An authentic cadence is a cadence with the progression V-I, or in this case A to D. This is an imperfect authentic cadence because neither the Soprano nor the Bass voices contain the root of the chord—in this case the Bass has the root, D, but the Soprano is on the third of the chord, F.

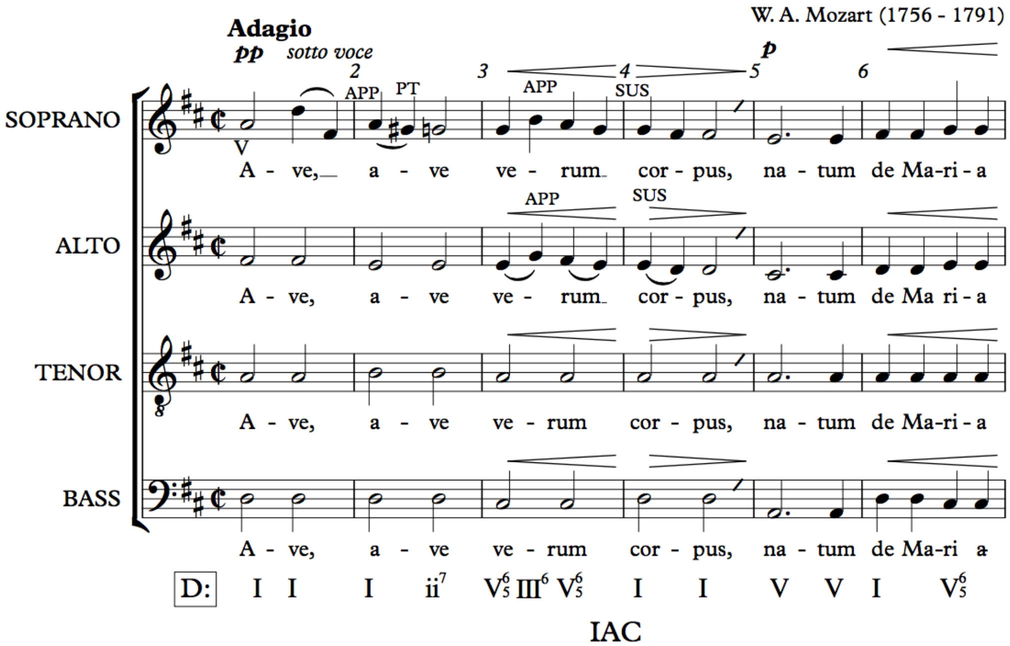


Figure 2-2

In mm. 7-8: bb. 3-1 there is a half cadence, or an "inconclusive cadence that ends on V" (see Figure 2-3 below) (Clendinning, p. 246).

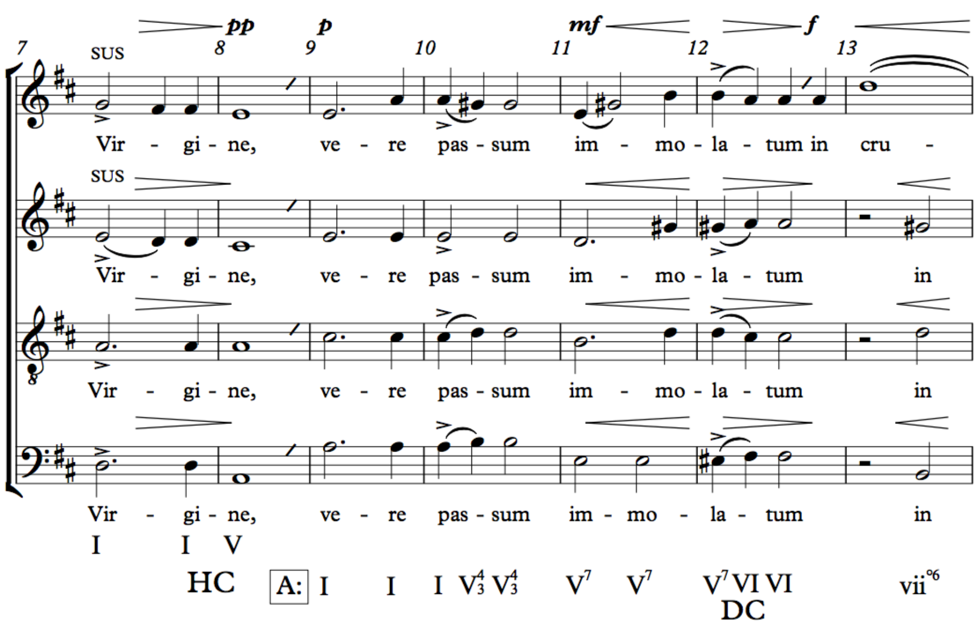


Figure 2-3

This is a model half cadence, with the dominant chord in root position and the fifth of the chord, in this case E, in the Soprano voice. While most other cadences are defined by their approaching and landing chords, half cadences are only defined by their landing chord of the dominant (Clendinning, p. 246).

There is then a deceptive cadence (DC) in m. 12: bb. 1-4 in the new key of A major (see Figure 2-3 above). A deceptive cadence has the progression V or V^7 to vi in major keys and the progression V or V^7 to VI in minor keys, or is, in general, “any nontonic resolution from V at a cadence” (Clendinning, p. A57).

The next cadence is a perfect authentic cadence (PAC) in mm. 15-16: bb. 3-1 in A major (see Figure 2-4 below).

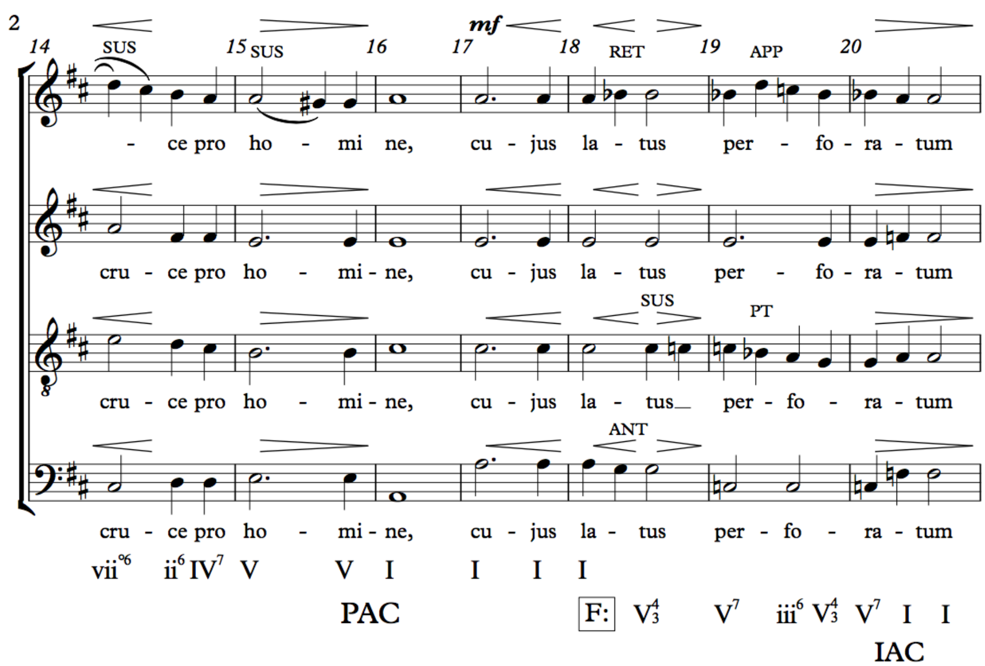


Figure 2-4

Additionally, a perfect authentic cadence requires that both the the Soprano and Bass are on the tonic (in this case A) in the final chord (Clendinning, p. 245). Note, the vii° chord may be substituted for the dominant chord and the cadence will remain authentic. This is the strongest type of cadence, and thus it is usually employed at the end of important phrases and as the final cadence in pieces. The next cadence is in mm. 19-20: bb. 4-4 in the new key of F major and is an imperfect authentic cadence, as seen before.

Following this there is a cadence in mm. 23-24: bb. 4-1 in d minor, and it is an half cadence (recall Figure 2-1, p. 7).

The next cadence is in mm. 31-32: bb. 4.5-1 and is back in the home key of D major. This is also a deceptive cadence like the previous cadence in A major, since here the progression is to V to IV (see Figure 2-5 below).

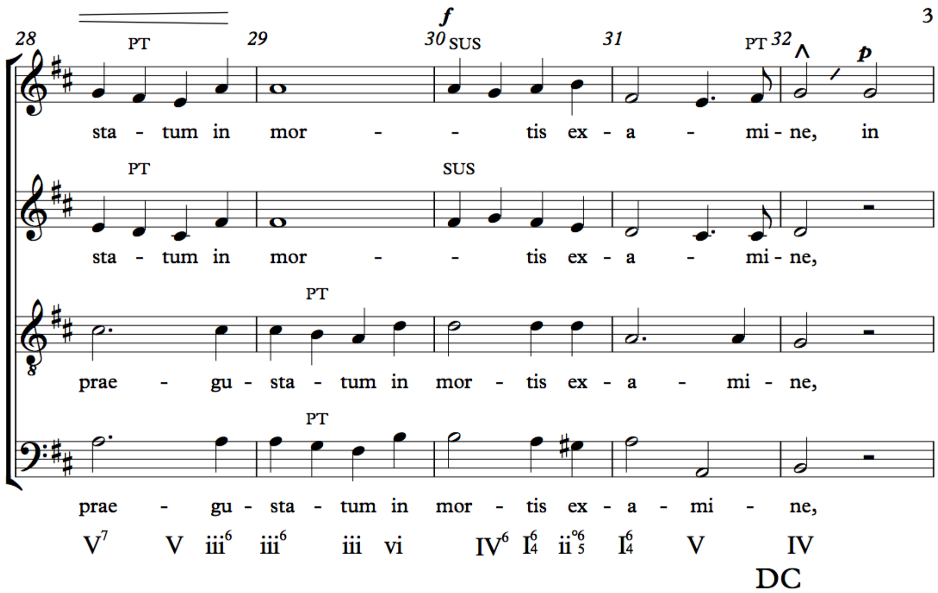


Figure 2-5

The final cadence of the piece is another perfect authentic cadence in the tonic key of D major in mm. 37-38: bb. 3-1 (see Figure 2-6 below).

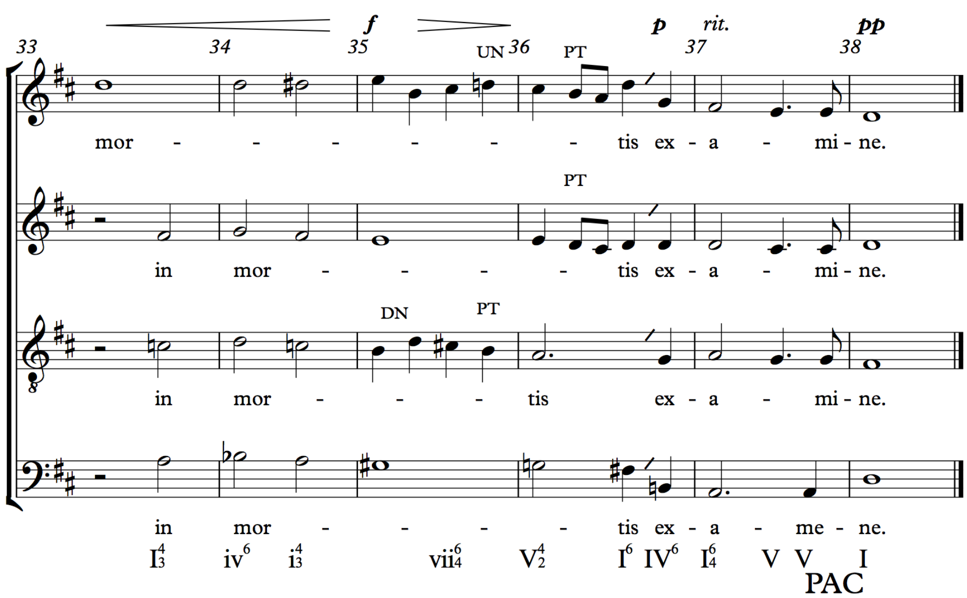


Figure 2-6

Here, the Soprano, Alto, and Bass are all on the tonic of D, and thus the ending cadence of the piece is the most conclusive. Notably, these cadences are often characterized by voives moving by whole step or common tone; movement by half-step is typically used for the leading tone, which is almost always present, and movement by leap is only used by the Bass voice. These consistencies will be important later in the analyses of our topologically transcribed pieces.

Non-Chord Tones

Another characteristic of Classical pieces is that any dissonances used are "approached and resolved within stylistic guidelines" (Clendinning, p. 213). For example, most dissonances occur during the use of non-chord tones such as appoggiaturas, suspensions, passing tones, and neighbor tones, such as in mm. 3-4: bb. 4-2 where the suspension on the first beat of the measure resolves appropriately down to a I chord on the second beat, as shown in Figure 2-7.

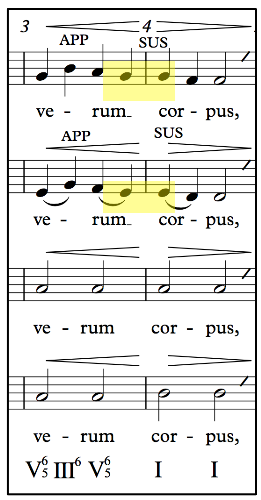


Figure 2-7

This is seen many times throughout the piece, and each of these embellishing tones has a specific approach and departure.

An appoggiatura is when the approach leaps to the dissonance and then resolves by stepwise movement in the opposite direction of the leap, as seen in mm. 1-2: bb. 4-2 in the Soprano part, m. 3: bb. 1-3 in the Soprano and Alto parts, and m. 19: bb. 1-3 in the Soprano part, as shown in Figure 2-8 (Clendinning, p. 328).

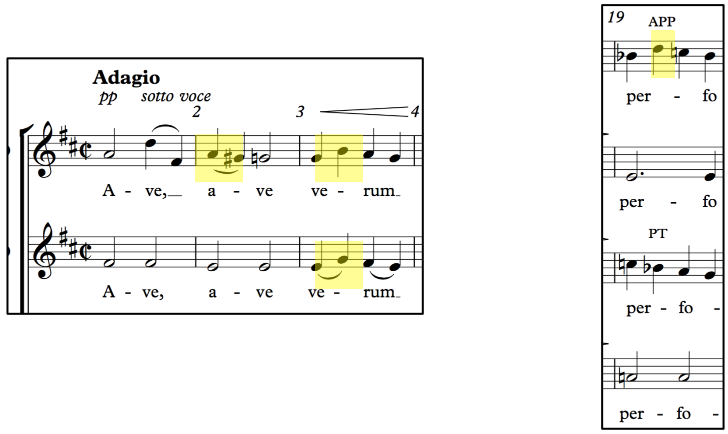


Figure 2-8

A suspension is “a dissonance configuration in which the dissonant or non-harmonic note is tied over from the previous beat (where it is consonant) and resolved by step, usually downwards” (Rushton). This is seen in mm. 3-4 in the Soprano and Alto parts (as seen before), mm. 13-14: bb. 1-1 in the Soprano, m.18: bb. 3-4 in the Tenor part, and mm. 14-15: bb. 4-3 in the Soprano part, as shown in Figure 2-9.

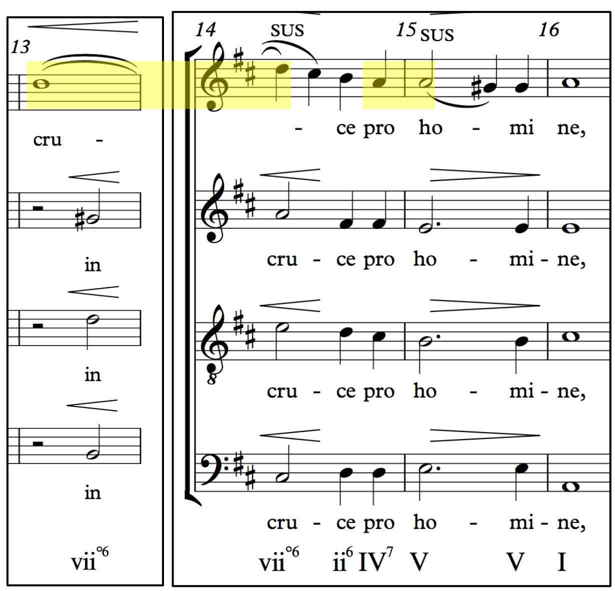


Figure 2-9

A passing tone is a non-chord tone that connects two notes by stepwise motion in either ascending or descending motion, as seen in m. 19: bb. 1-3 in the Tenor part, m. 27: bb. 1-3 and m. 29: bb. 1-3 in the Bass and Tenor parts, m. 31: b. 4.5 in the Soprano, and m. 36: b. 1-2 in the Soprano and Alto parts, as shown in Figure 2-10 (Clendinning, p. 189).

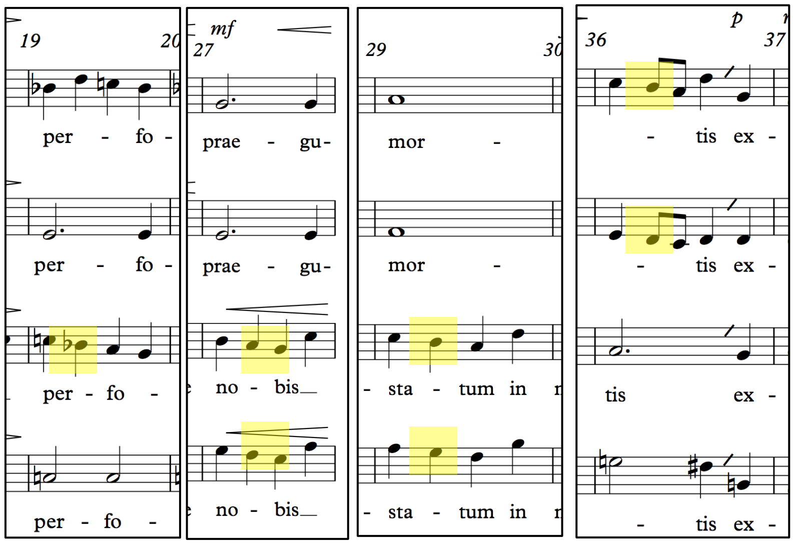


Figure 2-10

A neighbor tone is an “embellishment on the unaccented part of a measure that decorates a melody note by stepping to the note above or below it, then returning to the original note, as seen in m. 23: bb. 3-4 in the Soprano and Alto parts, as shown in Figure 2-11 (Clendinning, p. 191).

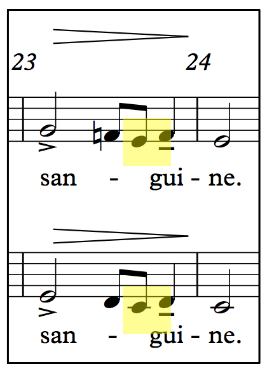


Figure 2-11

A double neighbor is “the combination of successive upper and lower neighbors (in either order) around the same pitch, as seen in mm. 34-35: bb. 1-2 in the Tenor, as shown in Figure 2-12 (Clendinning, p. A58).

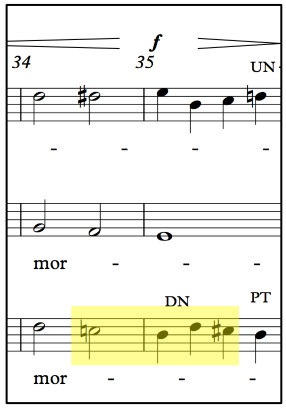


Figure 2-12

An anticipation is "a suspension whose non-harmonic note resolves upwards", as seen in m. 18, b. 2 in the Soprano and Bass voices, as shown in Figure 2-13 above (Rushton). Note, the non-chord tone is repeated in the next beat as a consonance.

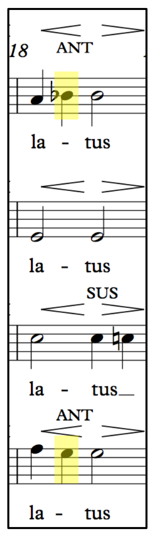


Figure 2-13

Thus, Mozart's *Ave Verum Corpus* exemplifies the characteristics of a Classical motet in four parts, and will provide a strong foundation for future comparison. When we transcribe the first new piece, we will be looking for characteristics similar to the ones exhibited here, that is, a clear key structure and means of modulation, the characteristics of the phrase melodies, the harmonic and melodic role of the pitches, the harmonic structure at cadence points, and the use, approach to, and departure from embellishing tones or dissonances.

3 Musical Analysis of *Ave Verum* Franz Liszt (1811-1886)

In order to confidently determine whether topological mapping has the ability to transcribe the harmonic characteristics of a Romantic motet, a piece that typifies such style must be used as the starting point. For this we will be using Franz Liszt's *Ave Verum*. The Roman-numeral analysis of the piece can be found on p. 73.

It is noted that both Mozart's and Liszt's pieces utilize similar text, however there is one key textural differences. The text Liszt uses is as follows:

"Ave verum corpus Christi	"Hail, true body of Christ
natum de Maria Virgine,	born of the Virgin Mary,
Vere passum immolatum	Who has truly suffered, was sacrificed
in cruce pro homine,	on the cross for mortals,
Cujus latus perforatum	Whose side was pierced
fudit aquam cum sanguine,	flooding water with blood,
Esto nobis praegustatum	Be for us a foretaste (of heaven)
mortis in examine.	During our final examining.
Amen."	Amen."
	Translation: Ron Jeffers

The only significant difference here is in line six, where "unda fluxit et sanguine" becomes "fudit aquam cum sanguine". Here, the meaning changes from "with water and blood" to "flooding water with blood", which increases the textual intensity. These changes were a result of a liturgical change in the Cistercian Rite. In addition to this change in wording, the Cistercian version of this prayer has an additional last phrase and some variants in which notes are assigned to which syllables (Joshua). The reason for these changes were that "the Cistercians rejected the elaborate liturgical practices of contemporary religious orders, in particular the liturgy of the Benedictine monks of Cluny… [and] sought to impose a liturgy that was simple and faithful to the *Rule of St Benedict*, stripp[ing] away appendages that had steadily accumulated over the centuries" (Kerr, p. 3). Thus, in using this version of *Ave Verum*, Liszt is using what some consider its purest form.

Furthermore, larger differences lie in musical expression and interpretation of the text. Unlike Classical music, music from the Romantic period is tumultuous and emotional, focusing on the meaning of the words in the text and often including ambiguous tonal centers and unclear cadences.

Like Mozart's earlier work, Liszt's *Ave Verum* is also homophonic and in four parts. However this piece is composed in a thoroughly Romantic idiom. For instance, this piece has an ambiguous tonal center, and even though Liszt outlines modulations to new keys, the tonal center rarely lies within the key specified. In fact, the only clear cadences throughout the whole piece are within the first phrase and at the end of the piece itself. This is best heard when listening to the piece in its entirety, for the listener finds some stability in the half cadence at mm. 4-5 and the plagal cadence at the end of the first phrase in mm. 9-10. However, there is no use of cadences after that as punctuation in any of the phrases until mm. 50-53, the last two measures of the piece (see Figures 3-1 and 3-2 below and Figure 3-6, p. 19).

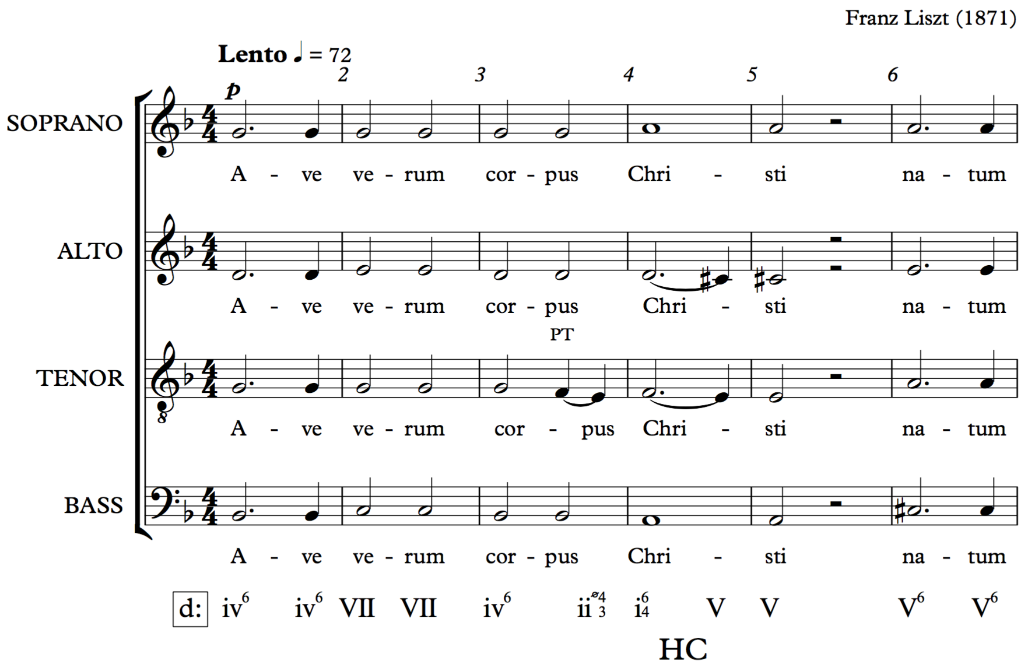


Figure 3-1

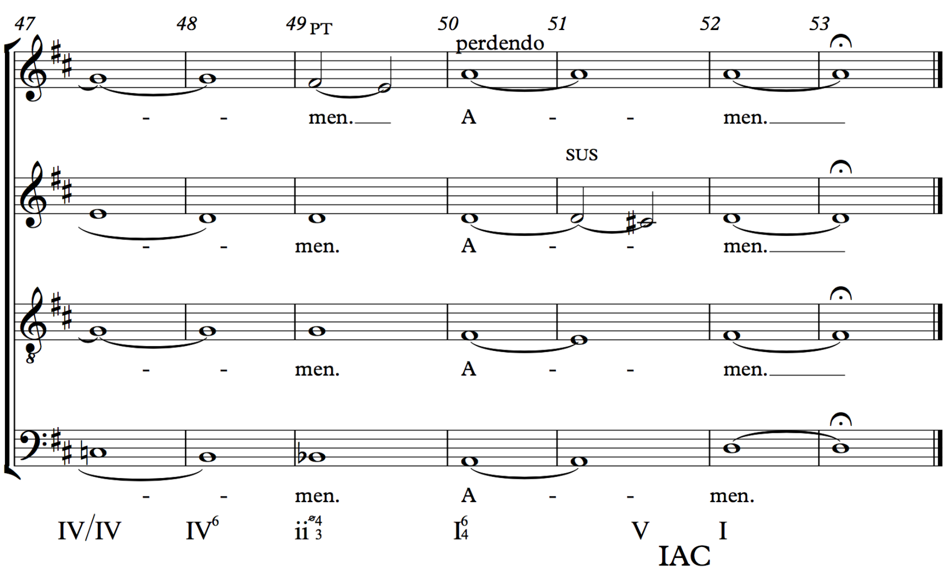


Figure 3-2

Although there is a half cadence due to the i-V movement in mm. 4-5, this cadence is not strong enough to punctuate the piece. On the other hand, another weak cadence employed here in mm. 9-10 is a plagal cadence (a cadence of the progression IV-I) that is most often used at the end of hymns on the “Amen” (Clendinning, p. 299). Here, its inconclusive nature adds to the tonal uncertainty of the piece while still signifying the end of the first phrase.

The final cadence in mm. 50-53 of the piece is an imperfect authentic cadence because the Bass contains the tonic of D while the Soprano ends on an A.

Thus the piece ends conclusively with an authentic cadence, but it still maintains some instability due to the imperfectness of the final cadence. This is Liszt’s technique throughout the entire piece, for he never allows the listener to feel “at home” in any key. This is especially apparent in mm. 31-45, where Liszt is in the key of D major, but never makes use of the tonic harmony. Furthermore, in this passage, he makes use of the Neopolitan 6th in mm. 39-40, a major II chord in first inversion based on the flatted second degree of the scale, in this case E-flat or D#; This chord functions primarily at cadences as “a chromatic alteration of a predominant harmony”, or in this case as a pre-dominant harmony before the end of a phrase lacking a cadence in order to extend the listener’s sense of imbalance (see Figure 3-3 below) (Clendinning, p. 531).

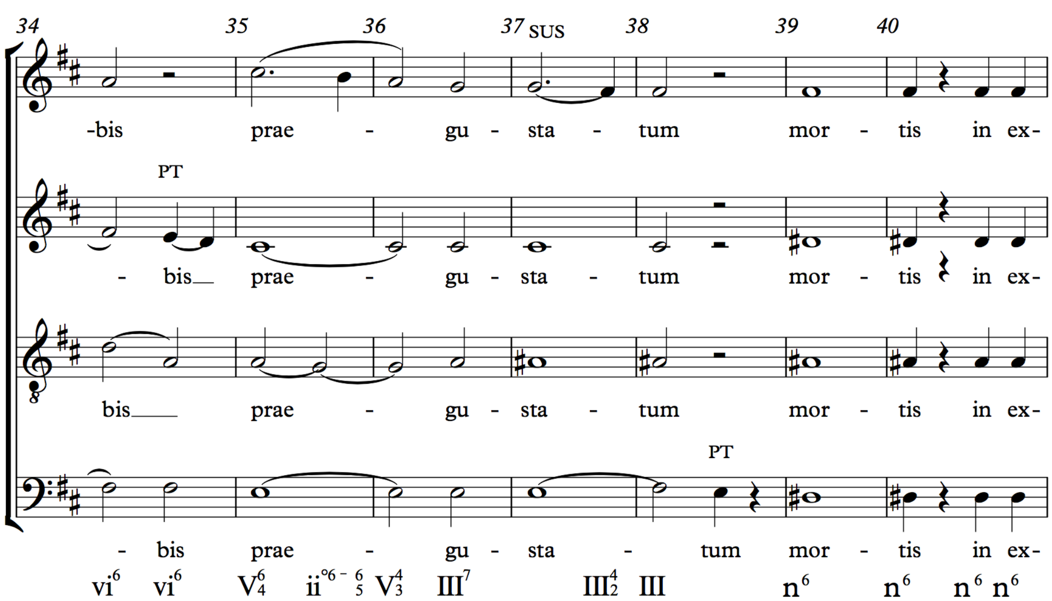


Figure 3-3

He then adds a seventh to the Neopolitan 6th in m. 41, a very unusual harmonic idea due to the fact that in addition to the suptertonic (ii) chord being major here when it is usually minor in the key of D major, however the seventh, C#, is the leading tone in the key of D-major, and thus it has theoretical value (see Figure 3-4 below). Thus this addition is a stylistic choice of Liszt's, and while it is unusual, adding to the ambiguity of the piece, it is theoretically sound.

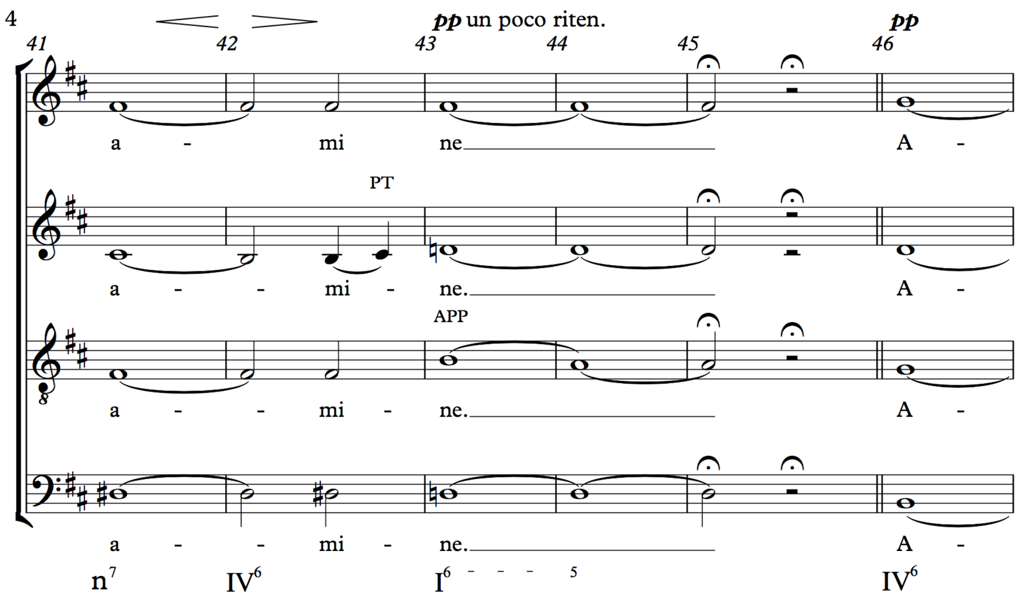


Figure 3-4

One of the most common harmonic progressions in a Romantic era piece is by circle-of-fifths. This is defined as when "dominant-seventh-quality chords are substituted for all or some…of the [successive] chords in the diatonic pattern", each root being a perfect fifth below that before it (Clendinning, p. 600). Liszt employs this progression for the use of modulation in mm. 20-29 of his piece, the descending circle-of-fifths sequence being a^7 (m. 23, b. 3) – d#^{o7} (m. 24, b. 3) – G#^{7} (m. 26, b. 1-2) – c# (m .26, b. 3-4) – F$^+$ (m. 27, b. 1-2), alternating with occasional B major chords and eventually ending with an inconclusive g#^{o7} or vii^{o7}/V chord in D major (see Figure 3-5 below). It is important to note however that the progression from a^7 to d#^{o7} is actually a descending tritone or diminished 5th and the progression from c# to F$^+$ is actually a descending augmented 5th. The use of diminished and augmented descending circle-of-fifths is not usual and contributes to the harmonic instability in this piece.

Figure 3-5

Note, the F^{+} chord is an augmented sixth chord in the key of a minor, a particularly unstable chord which is used to modulate back to the key of D major on a vii^{o7}/V chord in m. 27: b. 4 after a series of tumultuous modulations and tonicizations. The use of this modulation is called enharmonic modulation "because the harmonies resolve in ways that require reinterpretation" (Clendinning, p.631).

Furthermore, another common example of chromaticism in Romantic music—chromatic bass lines—accompanies this sequence. While the upper voices are completing the chords in the descending circle-of-fifths sequence, the Bass is using suspensions to travel stepwise downward from C to A in mm. 20-21:b. 1-3 and mm. 22-23: b. 1-3, and from C to G# in mm. 24-26: 1-1, as seen above in Figure 3-5.

This movement contributes to the modulation of the sequence through chromatic inflection, where the modulation is "activated by the shift of one pitch by a half step" (Clendinning, p. 625). This is seen in all of the voices in mm. 21, 23, 25-26, 27, and 28, where the piece is shifting rapidly between tonicizations in a minor, e minor, and c# minor, as seen above.

Furthermore, this piece doesn't have a part that dominates the melodic line. Throughout the first phrase the harmonic result is more important than the melodic. Then, the Bass begins the second phrase in m. 11: b. 1 in Figure 3-6 below and the other parts follow suit similarly, yet still none stand out melodically. This happens again in m. 20: b. 1, above in Figure 3-5.

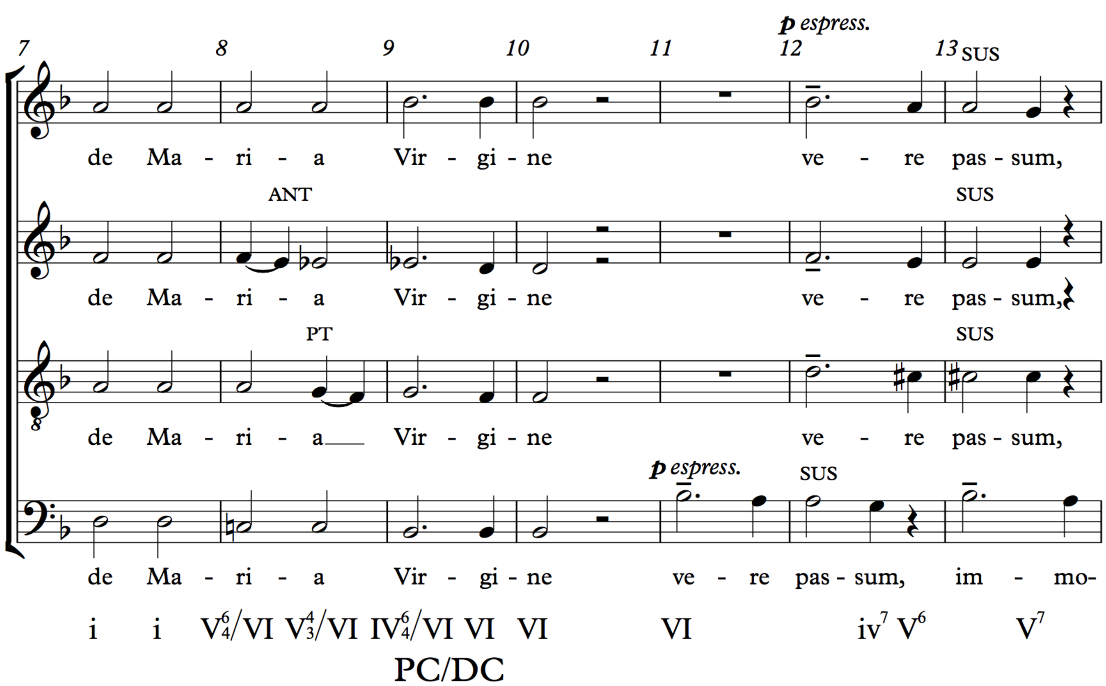


Figure 3-6

Finally, the last phrase is somewhat different in that the Soprano line from mm. 31-38, shown below, seems to follow a clear motivic pattern, however the lack of harmonic stability undermines this melodic idea, see Figure 3-7 below.

Figure 3-7

The final statement of the piece, the “Amen”, is sung by all voices with no one voice standing out as holding the melody. Thus, as is typical in Romantic music, all of the voices in this piece all equal in importance and work together to combine the harmonic, melodic, and emotional ideas in the idea of a resounding “amen”.

All of these musical elements are used in text painting, or as “a means of depicting images or ideas from a text through music (Clendinning, p. 558). The first line, “Ave verum corpus Christi natum de Maria Virgine”, or “Hail, true body of Christ born of the Virgin Mary”, is accompanied by a line with little movement, being divided by a half cadence and ending with a plagal cadence, which exposes the piece as a prayer rather than a song. The tone of the piece as set by this phrase is solemn and uncertain yet hopeful due to the final cadence in the major key of B.

The second line, “Vere passum immolatum in cruce pro homine”, or “Who has truly suffered, was sacrificed on the cross for mortals”, is the first line where embellishing tones are used for dissonances, and it is clearly a transitional phrase. It ends without a cadence, actually on a supertonic chord of the submediant (ii/vi) in m. 15: b. 3. This builds tension and leads clearly into the next phrase, landing on a V^7 chord in the key of b minor in m. 17: b. 1, continuing the tonicization of the submediant key.

The third phrase, perhaps the most textually alarming, is “Cujus latus perforatum fludit aquam cum sanguine”, or “Whose side was pierced flooding water with blood”. This phrase begins the same way as the second with the Basses cycling through suspensions, however this phrase is the most harmonically unstable and dissonant in the whole piece. Through the descending circle-of-fifths progression from a minor, e minor, and finally to c# minor, in addition to the movement by tritones, Liszt creates a harmonic representation of the pierced side of Christ due to the piercing nature of these diminished and augmented chords and their strangeness to the key of D major.

Finally, after a long pause, the last phrase prior to the closing “Amen” is “Esto nobis praegustatum mortis in examine”, or “Be for us a foretaste (of heaven) During our final examining”, which is asking for mercy and entrance to heaven. Supporting that idea, here is where the Soprano line ascends to its highest point in the whole piece, painting the idea of the height of heaven. And with the mention of death, all of the parts come together rhythmically in mm. 39-45 and in consonance in mm. 42-45, thus symbolizing togetherness in this uncontrollable fate.

Of the final two utterances of “amen”, the first is extremely dissonant, reflecting the trials of all of the words previously spoken. However the last is an oasis, and a symbol of hope at the end of this prayer, and thus coincides with the only harmonically clear and satisfying cadence in the entire piece. Additionally, the tessitura in which the voices are written—all of which are written lower in their ranges—making further reference to being on one’s knees praying to God.

Thus Liszt’s *Ave Verum* exemplifies characteristics of a Romantic motet in four parts, and will thus provide a second group of criterion for comparison. Upon the composition of the second new piece, we will be looking for characteristics similar to the ones exhibited here: use of ambiguous tonal center, modulatory melodic progressions, lack of clear cadences, and the uses of a descending fifth progression, chromaticism, and text painting.

4 Overview of Tymoczko's methods in "A Geometry of Music"

Before analyzing these pieces mathematically, it is important to understand how to geometrically map them and in what spaces they shall reside. Furthermore, it is then important to recognize what musical voice leading looks like in geometric chord space. Dimitri Tymoczko outlines the theory for such spaces in his book "A Geometry of Music". This will be our primary source for geometric and topological music analysis because it is the only source for this application of mathematics within music. This chapter paraphrases the work that Tymoczko has done in his book in order for the reader to understand our processes later on.

The space in which a chord lives depends greatly on the number of pitches in that chord. For instance, a chord with one pitch exists on a line, where we begin with C1, C#/D-flat1, and D1, and cycle through C4 all the way up to C7. To visualize this, we look at a number line with these associated values in Figure 4-1:

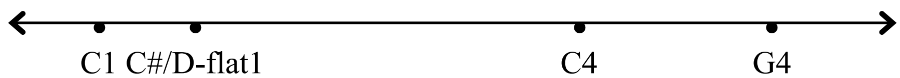


Figure 4-1

Now, although C1 and C4 are very different pitches due to their difference in octave, they are both members of the same pitch-class (pc). Thus, in order to reflect the inherent "equalness" between members of the same pc, we bend our line into a circle and eliminate the octave variable:

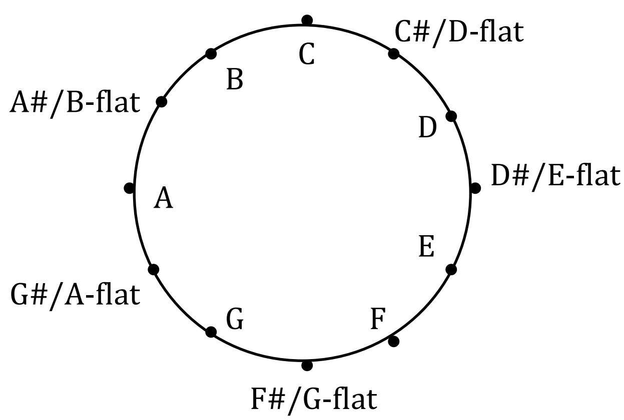


Figure 4-2

And thus we have created our space for one-note chords. Now, admittedly this space isn't very useful in terms of analyzing music, especially when considering voice leading, however some of the methods and ideas used to create this simple space are necessary in order to create and understand the space in which two, three, and four note chords live.

Moving on to two-note space, we now must account for two variables, and thus our line moves into Euclidean space. For example, if we were to map the ordered pair (C4, E4) we would need to find a point on the space such that C4 is represented on the x-axis and E4 is represented on the y-axis, as shown below.

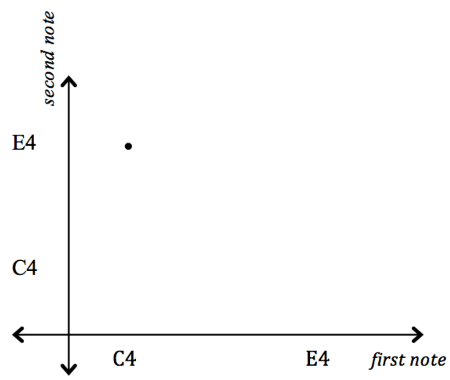


Figure 4-3
[Tymoczko, p. 66]

In Figure 4-3, we see a visual representation of the chord (C4, E4) in geometric space. This allows for a fairly simple visualization of two-note chords, however in this space there is still a distinction between C1 and C4. Like we did in our one-note chord space, this space must now be twisted to reflect pitch class rather than pitch only. Tymoczko does this first by writing out what each combination of two-note chords looks like with the distinction between pitches of similar pitch class and different octave, and then explains how to bend this space in order to eliminate this distinction. To begin, a more comprehensive portion of two-note chord space is shown in Figure 4-4.

Figure 4-4
[Tymoczko, p. 68]

As is apparent by this image, two-note chord space is rather complicated when octave is still a distinct variable. This is because, for this space to be best used to represent harmonic progression of two-note chords, the x and y axes as seen in Figure 4-3 have been rotated such that the x axis is now on the line $y = -x$ and the y axis is now on the line $y = x$. With that knowledge, this image becomes much more easy to understand.

Figure 4-5
[Tymoczko, p. 68]

However, taking away the octave variable, there is an apparent symmetry between these four quadrants of two-note chord space. Using the top right quadrant as our "standard" quadrant, we see that the symmetries between the four quadrants can be visualized as follows in Figure 4-5. Thus, if quadrants one, two, three, and four are labeled as such with our "standard" quadrant being quadrant one and moving clockwise, we see that quadrant two is a mirror image of quadrant one over the x-axis, quadrant four is quadrant one upside-down, and quadrant three is a mirror image of quadrant four over the x-axis.

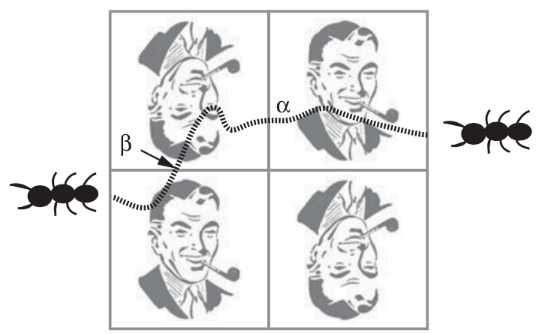


Figure 4-6
[Tymoczko, p. 69]

Now, in order to twist the space so that we are only looking at one quadrant and no longer considering octave, we turn to the "parable of the ant". Suppose that the path of an ant across this x-y space is as shown in Figure 4-6. In order to move this path onto our "standard" quadrant, we must consider the symmetries between our four quadrants. The ant begins by walking along quadrant one before crossing the y-axis into quadrant four, this crossing represented as α. It then travels through quadrant four before crossing the x-axis into quadrant three, this crossing represented by β. Now we consider what this path would look like only looking at our "standard" quadrant and using our knowledge of the symmetries between quadrants. The result is shown in Figure 4-7.

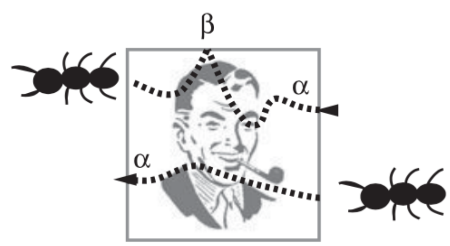


Figure 4-7
[Tymoczko, p. 69]

As can be seen, the ant's beginning path remains the same because it began in quadrant one, our "standard" quadrant. Then, at α, where the ant crossed into quadrant four, we see that it crossed onto the forehead of our image above the pipe, and thus we illustrate this by having alpha split off and re-appear at the top right of the quadrant, on the temple of the man above the pipe. Thus we see that crossings of the y-axis result in jumps from one side of our quadrant to the other, along with a twisted space in which the bottom left corner of the graph connects with the top right corner and the top left corner connects with the bottom right corner. Finally, because at β the ant traveled from the crown of our man's head to the crown of his head, we see that crossings of the x-axis result in our path "bouncing off" the edges of our graph, as also shown in Figure

4-7. Hence, our finalized two-note chord space in which octave is not a variable can roughly be visualized as shown in Figure 4-8, and is then written-out in Figure 4-9.Note that this space is commonly known as a Möbius strip, which will be explored later. Now, how space looks for three-note chords is shown in Figure 4-10, and because four-note chord space is abstract, as shown in Figure 4-11, we must use what we know about two and three-note spaces to work within it.

Figure 4-8
[Tymoczko, p. 70]

Therefore, it is apparent that the more notes in a chord, the more complicated the space in which that chord resides. However, for our purposes, there are similarities throughout all chord spaces that will allow us to have a method for mapping, inverting, and transcribing Mozart and Liszt's motets.

First, note that along the top and bottom of the two-note chord space lie chords that contain two of the same pitch class. Thus, whenever a composer moves from having four different pc to only three, which is common at cadences, topologically they are moving towards the edges of the space. Additionally movement along lines parallel to the y = -x axis results in one pitch remaining the same while the second moves up by half-steps, while movement along lines parallel to the y = x axis causes the second pitch to remain the same while the other pitch moves up by half step.

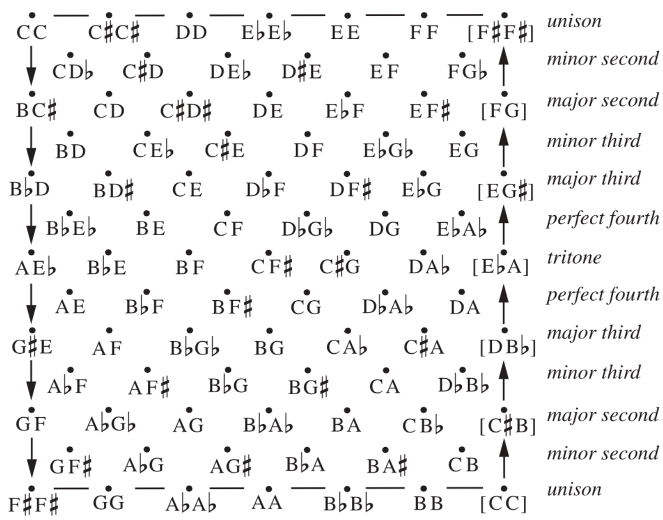


Figure 4-9
[Tymoczko, p. 71]

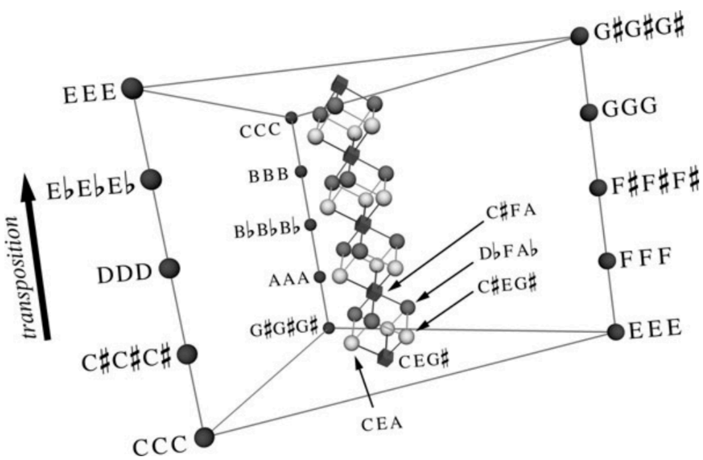


Figure 4-10
[Tymoczko, p. 86]

Similarly, in three-note chord space, chords that contain three of the same pitch are on one of the three edges of the space. However, in this space we look at "slices". For instance,in the top slice containing {C,C,C}, {G#,G#,G#}, and {E,E,E], as seen in figure 4-12, we see that movement from {C,C,C} to the opposite side leaves one pitch at C while the other two move by half steps in opposite directions. This is

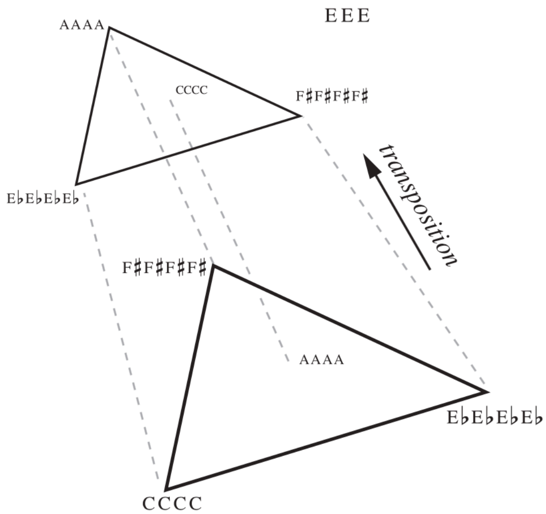


Figure 4-11
[Tymoczko, p. 94]

similar for the other two corners and their opposite sides. Hence, looking at "slices", we can thus map three-note chord space.

For our purposes, we will need to be even more creative in order to "map" Liszt's *Ave Verum* in four-dimensional chord space. Thus, we will be working with individual pitches in addition to groups of pitches in order to determine what inverting the set would do and where each voice would move in this inversion. To better prepare us, this chapter will end with an example of such a process.

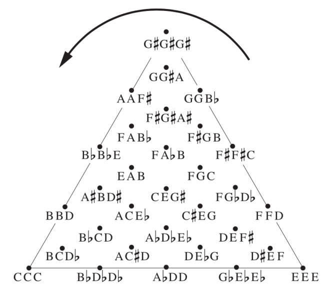


Figure 4-12
[Tymoczko, p. 89]

Example

What would inverting the voice leading from g^6 to C yield? In other words, how would we invert the path from {G,D,G,B-flat} to {G,E,G,C}?

First, notice that two pitches, in the same voices, remain the same. Thus our solution will contain two voices with those same pitches, i.e. {G,__,G,__}. Now, what is the voice leading from D to E? It is up by two half-steps, hence we will now move that pitch down by two half-steps, giving us {G,C,G,__}. Finally, since B-flat to C is a movement up by one half-step, we move down one half-step to get our final chord {G,C,G,A}.

We now have the tools and methods to analyze our two pieces and transcribe pieces in two, three, and quasi-four-dimensional space. Note that these methods to be used are created by me for the purpose of testing the validity of transcribing music using topology. However, before we begin our mathematical analysis and inversion of these pieces, in the next two chapters we will explore in more depth the mathematical spaces in which they exist in order to better predict what the results of our topological inversion will be.

5 Introduction to Orbifolds: Intro to Topology

In order to comprehend the space in which chords with four voices and thus pieces of music in four parts reside, a relative foundation of topology is first needed. In this chapter the relevant topics covered include: isometries, open sets, product topology, continuity, actions of topological groups, and properly discontinuous. To best illustrate these ideas while keeping the subject matter relevant, we will introduce these topics in relation the set representing general chord-space: $\mathbb{R}^n/G$. Note that this space needs further definition before being understood, which is done below. For now however, let it be assumed that this set makes sense for our purposes, and this fact will be proven through example over the next two chapters.

While $\mathbb{R}^n$ is a familiar set of the real numbers in *n* dimensions, in this case *G* is the set that's been defined to represent transpositions and permutations in this space in order to create chord space. That is,

$$G = \{\overline{\sigma}T_{\underline{k}} \mid \overline{\sigma} \in \overline{S_n}, T_{\underline{k}} \in N\}, \text{ where}$$

$$N = \{T_{\underline{k}} \mid k \in \mathbb{Z}^n\}$$

$$\text{and } T_{\underline{k}} : \mathbb{R}^n \to \mathbb{R}^n \text{ is the map where}$$

$$T_{\underline{k}}(x_1, x_2, \dots, x_n) = (x_1 + 12k_1, x_2 + 12k_2, \dots, x_n + 12k_n)$$

$$\text{for } \underline{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n,$$

and where $\overline{S_n}$ acts on $\mathbb{R}^n$ by permuting the coordinates.

Thus, we can see that $\overline{\sigma}$ acts on a space by permuting the given coordinates, and $T_{\underline{k}}$ does so by adding 12 to the given coordinates, or transposing in this case. Because chord-space, a subset of pitch-class space, is the set of all combinations of pitches from a set of 12, using modular arithmetic and this set we can relate numbers to pitches and thus have a space with all possibilities for permutations and transpositions. Now, this set will be used as a basis of discussion for the aforementioned topics.

Isometries

First, for our purposes the elements of *G* need to be isometries. The correct setting for isometries is within a metric space.

Definition 1. *A **metric** on a set X is a function*

$$d: X \times X \to R$$

having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$.
(3) (Triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$ *for all* $x, y, z \in X$.

(Munkres, p. 117)

Definition 2. *A* ***metric space*** *is a [set] X together with a specific metric d…*

(Munkres, p. 119)

From this we can see that the elements of *G*, the way that it has been defined, are isometries.

Definition 3. *A continuous map* $f: X \longrightarrow X$ *is an* ***isometry*** *if* $d\big(f(x), f(y)\big) = d(x,y)$ *for all points* $x, y \in X$ (Munkres, p. 182).

With this definition, it is clear that neither adding a constant to each pitch nor switching their order will change the distance between them, thus confirming that the set *G* is a set of isometries.

Additionally, it can also be shown that the group of all isometries of $\mathbb{R}^n$ is a (infinite) group through composition. The set *G* is a subgroup of this group.

Proof. To prove *G* is a subgroup of the group of all isometries of $\mathbb{R}^n$ we must show that $\bar{\sigma}T_{\underline{k}}\bar{\tau}T_{\underline{l}} \in G$ for $\bar{\sigma}T_{\underline{k}}, \bar{\tau}T_{\underline{l}} \in G$ and $(\bar{\sigma}T_{\underline{k}})^{-1} \in G$ for all $\bar{\sigma}T_{\underline{k}} \in G$. Thus, we first have:

$$\bar{\sigma}T_{\underline{k}}\bar{\tau}T_{\underline{l}} = \bar{\sigma}\bar{\tau}(\bar{\tau}^{-1}\, T_{\underline{k}}\bar{\tau})\, T_{\underline{l}}.$$

Now, to continue, we show $\bar{\tau}^{-1}\, T_{\underline{k}}\bar{\tau} = T_{\bar{\tau}^{-1}(\underline{k})}$:

$$\begin{aligned} \bar{\tau}^{-1}\, T_{\underline{k}}\bar{\tau}(v) &= \bar{\tau}^{-1}\,(\bar{\tau}(v) + \underline{k}) \\ &= \bar{\tau}^{-1}(\bar{\tau}(v)) + \bar{\tau}^{-1}(\underline{k}) \\ &= v + \bar{\tau}^{-1}(\underline{k}) \\ &= T_{\bar{\tau}^{-1}(\underline{k})}(v). \end{aligned}$$

Thus, since $\bar{\tau}^{-1}\, T_{\underline{k}}\bar{\tau} = T_{\bar{\tau}^{-1}(\underline{k})}$, we have

$$\bar{\sigma}T_{\underline{k}}\bar{\tau}T_{\underline{l}} = \bar{\sigma}\bar{\tau}T_{\bar{\tau}^{-1}(\underline{k})}T_{\underline{l}}$$

where $\bar{\sigma}\bar{\tau} \in \overline{S_n}$ and $T_{\bar{\tau}^{-1}(\underline{k})}T_{\underline{l}} \in N$.
Next, suppose $\bar{\sigma}T_{\underline{k}} \in G$.
Then $(\bar{\sigma}T_{\underline{k}})^{-1} = {T_{\underline{k}}}^{-1}\bar{\sigma}^{-1} = T_{\underline{-k}}\bar{\sigma}^{-1} = \bar{\sigma}^{-1}\big(\bar{\sigma}T_{\underline{-k}}\bar{\sigma}^{-1}\big)$.

The fact that ${T_{\underline{k}}}^{-1} = T_{\underline{-k}}$ is clear since $T_{\underline{k}}(v) = (v + k)$, of which it is clear the inverse would be $(v - k) = T_{\underline{-k}}$.
Now, similarly to before, we have

$$\begin{aligned}\bar{\sigma}T_{\underline{-k}}\bar{\sigma}^{-1}(v) &= \bar{\sigma}(\bar{\sigma}^{-1}(v) - \underline{k}) \\ &= \bar{\sigma}(\bar{\sigma}^{-1}(v)) - \bar{\sigma}(\underline{k}) \\ &= v - \bar{\sigma}(\underline{k}) \\ &= T_{-\bar{\sigma}(\underline{k})}\end{aligned}$$

Hence $\bar{\sigma}^{-1}\left(\bar{\sigma}T_{\underline{-k}}\bar{\sigma}^{-1}\right) = \bar{\sigma}^{-1}T_{-\bar{\sigma}(\underline{k})} \in G$ since $\bar{\sigma}^{-1} \in \overline{S_n}$ and $T_{-\bar{\sigma}(\underline{k})} \in N$.
Therefore we have proved our claim. □

Open Sets

Now, to further understand this chord-space-set it is important to understand the "topology" of this set. Essentially, that is what the set "looks like" and how the points within it are characterized. First, it will be shown that the set $\mathbb{R}^n/G$ is an open set.

Definition 4. *A* ***topology*** *on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

(1) $\emptyset$ and X are in $\mathcal{T}$.
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(Munkres, p. 76)

Definition 5. *If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an* ***open set*** *of X if U belongs to the collection $\mathcal{T}$ ... [Thus] a topological space is a set X together with a collection of subsets of X, called 'open sets', such that $\emptyset$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.*

(Munkres, p. 76)

Definition 6. *A set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$* (Munkres, p. 118).

In this case, think of $X = \mathbb{R}^2$, $\mathcal{T}$ as the collection of all open sets, and B_d as the open disk with center open disk with center *y*. Now, it is clear that the "topology" of $\mathbb{R}^2$ is that of the Cartesian plane, that is, that the distance between two points *x* and y follows: $d(x, y) = \sqrt{(x - x_1)^2 + (y - y_1)^2}$. Thus a topology must be assigned to the group *G*. Put a topology on *G* where every set is open. Then, when is a subset of $\mathbb{R}^n$ open? Before answering this, a few more topics need to be covered. This will be answered in the section on Product Topology, as will the next logical question, that is, when is the chord-space-set open? These two questions go hand-in-hand.

Product Topology

Because this chord-space-set is a product of two sets, in order to see how the preceding and following topics apply to this chord-space, it is important to understand product topology.

Definition 7. *If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that*

1) *For each $x \in X$, there is at least one basis element B containing x*
2) *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

(Munkres, p. 78)

Definition 8. *If $\mathcal{B}$ is a basis for a topology on X, the **topology generated by** $\mathcal{B}$ is described as follows: A subset U of X is said to be open in X (that is, to be an element of) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.*

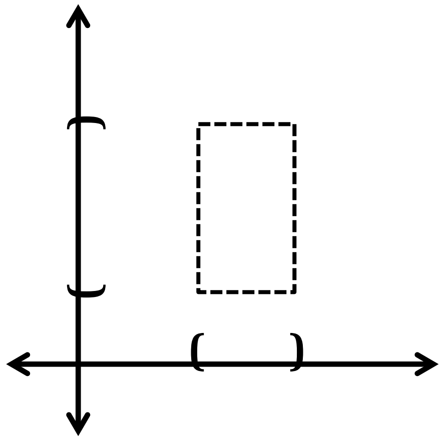
(Munkres, p. 78)

Definition 9. *Let X and Y are topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and Y is an open subset of Y* (Munkres, p. 86).

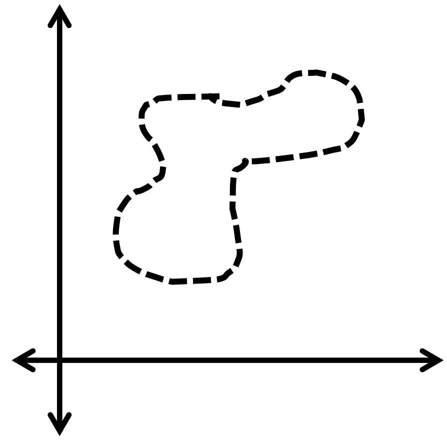
Now, a logical question would be, what does this set look like? While the next chapter will enter into more specifics on that question, in order to understand that the topology of this set must first be understood and assigned.

Here, think of $X = \mathbb{R}^2$ and $\mathcal{B}$ as the collection of all open disks. From this, it follows that open sets in $X \times Y$ are unions of sets of the form $N = U \times V$ where U is open in X and V is open in Y. Hence, it can be seen that since every set in G is open due to the topology it has been assigned, the chord-space-set $\mathbb{R}^n/G$ will be open whenever the set in $\mathbb{R}^n$ is open, answering our second question from the section on Open Sets. Now, in order to better define this, we must answer the first question from the section on Open Sets. That is, when is $\mathbb{R}^n$ open? To answer this, observe the case of $\mathbb{R}^2$.

Example 1. Clearly, $\mathbb{R}^2$ contains open sets. However, there are two cases on how to find them using our idea of topology:



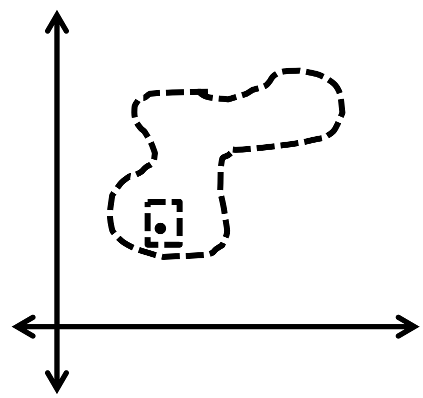
Case 1



Case 2

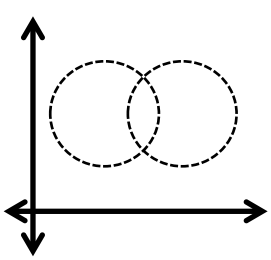
In Case 1, it is clearly seen that the space is open since it is clearly the product space of two open sets, hence, through product topology, this set is open in $\mathbb{R}^2$. However, Case 2 is clearly different. For the purposes of this example the given set is known to be open, but how do we prove that fact?

Consider the points in this set. Since we know that this set is open, any individual point can be found to be in a smaller set that is a product of two open sets, similar to Case 1, simply smaller. This is shown in Case 3. This can then be done for all of the points in this set, thus providing a method of proving that the set in Case 2 is open.



Case 3

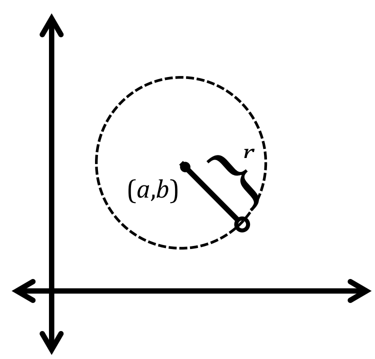
Example 2. Thinking of this basis as disks, the intersection of two open disks is an open set. To prove this, we use a similar methodology as in Example 1.



Case 5

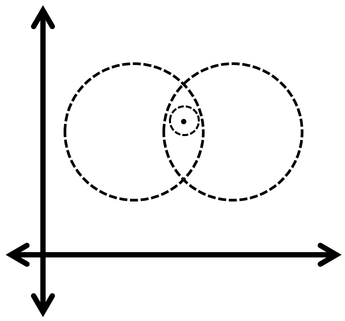
First, an open disk with center (a, b), radius r is

$$\{(x, y) \in \mathbb{R}^2 |\ d\big((x, y), (a, b)\big) < r\}.$$



Case 4

This would look like what is seen in Case 4. Now, let's look at the intersection of two open disks, like in Case 5. In this case, for each point in the intersections of these two



Case 6

sets we can find a smaller disc containing this point such that this smaller disk is open, as shown in Case 6. Thus, through this method in conjunction with that used in Example 1 we can find all open sets in $\mathbb{R}^2$.

Now, throughout this chapter there have been two kinds of topology assigned to our set: metric topology and product topology. While these may seem different, especially due to their difference in nomenclature, surprisingly here in both $\mathbb{R}^2$ and in $\mathbb{R}^n$ metric topology and product topology are the same.

Continuity

The next question needed to further understand our set is: is the mapping $\pi\colon G \times \mathbb{R}^n \to \mathbb{R}^n$ continuous? Note that this mapping is defined by $\pi(g, \underline{v}) \mapsto g \cdot \underline{v}$.

Definition 10. *Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be* ***continuous*** *if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

(Munkres, p. 102)

Thus, with this definition, the question becomes: When O is open in $\mathbb{R}^n$, is $\pi^{-1}(O)$ open? To answer this, we have the following considerations:

$$\begin{aligned} \pi^{-1}(O) &= \{(g, \underline{v}) \mid \pi(g, \underline{v}) \in O\} \\ &= \{(g, \underline{v}) \mid g \cdot \underline{v} \in O\} \\ &= \bigcup_{g \in G} \{g\} \times g^{-1}(O) \end{aligned} \tag{1}$$

which is open because all sets in G are open and O is open in $\mathbb{R}^n$ by our assumption. Hence, what's above proves that the mapping $\pi\colon G \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Actions of Topological Groups

Now, in the chord-space-set $\mathbb{R}^n/G$, we will show that G is acting on $\mathbb{R}^n$. To do this, we must first see that G is a topological group.

Definition 11. ***Topological groups*** *have the algebraic structure of a group and the topological structure of a topological space and they are linked by the requirement that multiplication and inversion are continuous functions* (Hofmann, p. 1).

This was effectively proven when *G* was proven to be an isometry.

Definition 12. *An* ***action*** *of a topological group G on a space X is a continuous map* $G \times X \to X$*, denoted by* $(g, x) \mapsto gx$*, so that* $g(hx) = (gh)x$ *and that* $1x = x$ (*Introduction to Orbifolds*, p. 2).

The work in (1) on p. 33 proves that the map of G on the space $\mathbb{R}^n$is continuous, and hence shows that the group G on $\mathbb{R}^n$ satisfies the criterion in Definition 12. Thus it is clear that G is a topological set acting on $\mathbb{R}^n$. Another definition that is associated with Definition 12 that will be useful later on that in Definition 13.

Definition 13. *The* ***isotropy group*** *of* $s_0 \in S$ *consists of the subgroup of G which does not move* s_0*; thus* $G_{s_0} = \{g \in G | gs_0 = s_0\}$ (Montgomery, p. 1).

The importance of G being a topological set that is acting on $\mathbb{R}^n$ in our chord-space set becomes apparent in the final section of this chapter.

Properly Discontinuous

Finally, we will see that the chord-space set, where the topological space *G* is acting on $\mathbb{R}^n$, is properly discontinuous.

Definition 14. *If* Γ *acts on a space X, this action is said to be* ***properly discontinuous*** *if, given two points* $x, y \in X$ *there are open neighborhoods U of x and V of y for which* $(\gamma U) \cap V \neq \emptyset$ *for only finitely many* $\gamma \in \Gamma$ (*Introduction to Orbifolds*, p. 2).

In our case, $\Gamma = \mathrm{G}$, $X = \mathbb{R}^n$, U and V are open sets containing $x, y \in \mathbb{R}^n$, and $\gamma = g \in G$. To prove that our set is properly discontinuous, we must show that there are only finitely many $g \in G$ such that $(gU) \cap V \neq \emptyset$.

Proof.
First, consider $X = \mathbb{R}$ and $x, y \in U$ where $U = (x - 1, x + 1)$ and $V = (y - 1, y + 1)$.
Hence $(U) \cap V \neq \emptyset$ when $d(x, y) < 2$.
Now, $g(x) = x + 12k$ for $k \in \mathbb{Z}$.
Then $gU = (g(x) - 1, g(x) + 1) = (x + 12k - 1, x + 12k + 1)$.
Thus for $(gU) \cap V \neq \emptyset$, we need $d(x + 12k, y) < 2$.
Clearly for this, there is only one value of k, since every G adds a multiple of 12 to x.

Thus for every $x, y \in \mathbb{R}^n$, there are only finitely many, in fact there is only one, $g \in G$ such that $(gU) \cap V \neq \emptyset$.

Now, for $\mathbb{R}^n$ where $n \geq 2$, first notice that permuting values does not change their distance, hence to prove that there are only finitely many $g \in G$ such that $(gU) \cap V \neq \emptyset$ in this case would be enough to prove our claim.

For each $x, y \in \mathbb{R}^n$ where $n \geq 2$, we have $(x_1, x_2, \dots, x_n) \in U$ and $(y_1, y_2, \dots, y_n) \in V$, where U and V are defined similarly to before.

Thus $gx = (x_1 + 12k_1, x_2 + 12k_2, \dots, x_n + 12k_n)$ for $\underline{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$.

From this, we see that $(gU) \cap V \neq \emptyset$ when $d(x_1 + 12k_1, y_1) < 2$, $(x_2 + 12k_2, y_2) < 2$, …, and $(x_n + 12k_n, y_n) < 2$.

Again, this is only true for one value of $k_1, k_2, \dots, k_n$.

Thus, there are only finitely many $g \in G$ such that $(gU) \cap V \neq \emptyset$ in this case.

Therefore, $\mathbb{R}^n/G$ is properly discontinuous. □

Thus, we have proven that $\mathbb{R}^n/G$ is a properly discontinuous set where G is a topological set that is acting on $\mathbb{R}^n$. This space thus defines general chord space. In the next chapter, we will explore further the properties of this set and thus what they suggest about the potential results of our future topological mappings.

6 Introduction to Orbifolds: Where Chords Reside

To understand the complexity of an orbifold, it is best to understand how points within the set are organized and how they function. As we already know, our set $\mathbb{R}^n/G$, where G is acting on $\mathbb{R}^n$, is properly discontinuous. To understand how the points in this set function, and more specifically, to understand how points function when G is acting on $\mathbb{R}^n$, we need a few more definitions.

Definition 1. *An* ***equivalence relation*** *on a set A is a relation C on A having the following three properties:*

(1) (Reflexivity) $x \sim x$ for every x in A.
(2) (Symmetry) If $x \sim y$, then $y \sim x$.
(3) (Transitivity) If $x \sim y$ and $y \sim z$, then $x \sim z$.

(Munkres, p. 22)

For example, consider the relation $\sim$ on $\mathbb{R}^n$ where $x \sim y$ if there exits $g \in G$ such that $y = gx$. Remember, G is our set of permutations and octave shifts. We claim that this is an equivalence relation.

Proof.

(1) Since G is the set of permutations and octave shifts, $x = gx$ when g is the identity.
(2) Suppose $x \sim y$. Then $y = gx$ for some $g \in G$. Then $yg^{-1} = x$, and $g^{-1} \in G$ since G is a subset of the set of all isometries. Thus we have $y \sim x$.
(3) Suppose $x \sim y$ and $y \sim z$. Then there exist $g_1, g_2 \in G$ such that $y = g_1 x$ and $z = g_2 y$. Then $z = g_2(g_1 x) = (g_2 g_1)x$, where $g_1 g_2 \in G$, again since G is a subset of the set of all ismoetries. Thus we have $x \sim z$.

Hence $\sim$ is an equivalence relation. □

Now that we have this, we can group the elements that are related by equivalence relations into sets called equivalence classes.

Definition 2. *Given an equivalence relation $\sim$ on a set A and an element x of A, we define a certain subset E of A, called the* ***equivalence class*** *determined by x, by the equation*

$$E = \{y \mid y \sim x\}.$$

(Munkres, p. 22)

It is known that the collection of all of the equivalence classes of a set is equal to the set itself, and hence we have that $\mathbb{R}^n/G$ is the set of equivalence classes. Now, define a function

$$\pi: \mathbb{R}^n \to \mathbb{R}^n/G$$

where $\pi(x) =$ the equivalence class containing x.

Before, when putting a topology on a set, we were only looking at one topological space. However, now, in order to put a topology on $\mathbb{R}^n/G$, we must understand quotient topology. In order to do this, we must first look at what a quotient map is.

Definition 3. *Let X and Y be topological spaces; let $p\colon X \to Y$ be a surjective (p maps X onto Y) map. The map p is said to be a* ***quotient map****, provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X (Munkres, p. 135).*

Note: this definition is similar to that of an open set. It is from here that we can get our next definition on quotient topology.

Definition 4. *If X is a space and A is a set and if $p\colon X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; it is called the* ***quotient topology*** *induced by p (Munkres, p. 136).*

Thus, if we give $\mathbb{R}^n/G$ the quotient topology, then our function π is a quotient map by this definition. Knowing all this, and connecting it to what we already knew about our chord-space set, we have the following theorems. Before this Theorem however, we need one more definition.

Definition 5. *Let X and Y be topological spaces; let $f\colon X \to Y$ be a bijection. If both the function f and the inverse function*

$$f^{-1}\colon Y \to X$$

are continuous, then f is called a ***homeomorphism*** (Munkres, p. 104).

Theorem 1. Consider the set where G is acting on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Then, if for the isotropy group G_x, if $|G_x| = 1$, there exists an open set $U \subseteq \mathbb{R}^n$ such that $x \in U$ and the restriction of π defined by

$$\pi_U\colon U \to \pi(U)$$

is a homeomorphism. In particular, there is an open set $(\pi(U))$ containing $\pi(x)$ that is homeomorphic to an open subset of $\mathbb{R}^n$.

Proof.
First, since π_U is a quotient map, we see that π_U is continuous.
Is it one-to-one?
Consider an arbitrary $x \in U$.
Now suppose for the sake of contradiction that $U \cap gU \neq \emptyset$.
Then, for $x \in U$, we have $x = gx$.
But since $|G_x| = 1$, g must be the identity function.
Hence, if $g \neq$ identity, we have $U \cap gU = \emptyset$.
Thus π is one-to-one.
Finally, does π_U map onto $\pi(U)$?
Let $y \in \pi(U)$.
Then $y = gx$ for some $g \in G$ and some $x \in U$.
Thus π_U maps onto $\pi(U)$.
Now, we must check that ${\pi_U}^{-1}$ is continuous.
This is again clear from the definition of quotient topology.
Hence π_U is a homeomorphism. □

Now, this theorem connects to our next theorem, which will be considered in the case of $\mathbb{R}^2$. However, before the statement of this proof, it is important to understand the relationship between $\mathbb{R}^2$, the Möbius band $\mathbb{R}^2/G$ (as mentioned in Chapter 4), and $\mathbb{R}^2$ modulo G_x, which is visualized below in Figure 6-1.

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[d, "\alpha"'] \arrow[drr, "\pi"] & & \\
\mathbb{R}^2/G_x & & \mathbb{R}^2/\mathrm{G}
\end{tikzcd}
$$

Figure 6-1

Note that this diagram is unmistakably similar to the diagram for the First Isomorphism Theorem of Group Theory.

Extra Theorem. (First Isomorphism Theorem). Let $\phi: G \to G'$ be a homomorphism of groups. Suppose that ϕ is onto and let *H* be the kernel of ϕ. Then *G'* is isomorphic to *G/H* (McKernan, p. 1).

The diagram for this theorem is as follows:

$$
\begin{tikzcd}
G \arrow[d, "\phi"'] \arrow[drr, "\pi"] & & \\
G' \arrow[rr, "f"'] & & G/K
\end{tikzcd}
$$

Figure 6-2

This diagram is clearly similar to that in Figure 6-1. Hence, there is a clear connection to make between our Theorem A, to be stated on p. 38, and this First Isomorphism Theorem.

Now, it is clear that in Theorem 1, because $|G_x| = 1$ and thus $G_x = \{identity\}$, $\mathbb{R}^2/G_x = \mathbb{R}^2$. Thus, Theorem 1 could also be stated: "In particular, there is an open set $(\pi(U))$ containing $\pi(x)$ that is homeomorphic to an open subset of $\mathbb{R}^n/G_x$. However, when $|G_x| = 2$, the relationship $\mathbb{R}^2/G_x = \mathbb{R}^2$ does not hold. Thus we instead have Theorem 2 in the case of $\mathbb{R}^2$ with $|G_x| = 2$.

Theorem 2. Consider the set where G is acting on $\mathbb{R}^2$. Let $x \in \mathbb{R}^2$. Then, for the isotropy group G_x, if $|G_x| = 2$, there exists an open set $U \subseteq \mathbb{R}^2$ such that $x \in U$ yields the restriction of π defined by

$$\pi_U : U \to \pi(U).$$

In particular, there is an open set $(\pi(U))$ containing $\pi(x)$ that is homeomorphic to an open subset of the half-space $H = \{(x_1, x_2) | x_1 \geq x_2\}$, i.e. ρ_U defined by

$$\rho_U : \pi(U) \to H$$

is a homeomorphism.

Proof.
Again, since ρ_U is a quotient map, we see that both ρ_U and ${\rho_U}^{-1}$ are continuous.
Is it one-to-one?

Let $x \in U$.
Pick U_x such that $U_x \cap gU_x = \emptyset$ unless if $g \in G_x$.
Now, consider an arbitrary $a \in \pi(U_x)$.
Then $a = (m, n)$ for $m, n \in \mathbb{R}$ with $m \geq n$.
Also, for $g \in G_x$, we have $a = ga$.
Since $|G_x| = 2$, g is not just the identity
Then $m = n$ and thus $(m, n) = (n, m)$.
Thus if $b \in H$, i.e. $b = (x_1, x_2)$ such that $x_1 \geq x_2$ and $b = a$ we have $(x_1, x_2) = (m, m)$, thus $x_1 = x_2 = m = n$ and ρ_U is one-to-one in this case.
Thus ρ_U is one-to-one.
Now, does ρ_U map onto H?
Let $y \in H$
Then $a = (x, y)$ for $x \geq y$.
Let g be the identity, thus $g \in G$, and let $a = b$ (note $a \in U$).
Then $b = ga \in \pi(U)$.
Thus $\pi(U)$ maps onto H.
Hence π_U is a homeomorphism. □

Note that $\mathbb{R}^2/G_x$ is homeomorphic to H, and thus by proving that $(\pi(U))$ is homeomorphic to H, and thus by proving this we have proven that $(\pi(U))$ is homeomorphic to $\mathbb{R}^2/G_x$, that is, that the function α_U defined by

$$\alpha_U : \pi(U) \to \alpha(U)$$

is a homeomorphism.

Furthermore, this Theorem can be expanded to more theorems regarding the cases where $|G_x| = 6$ and $|G_x| = 24$, each being in the spaces $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively. Note also that with these expansions that all of the previous theorems hold, that is, in $\mathbb{R}^3$ Theorems 1, 2, and 3 would hold, and in $\mathbb{R}^4$, Theorems 1, 2, 3, and 4 would hold. Finally, from this pattern, it can be seen that for $\mathbb{R}^n$, $|G_x| = n!$ and Theorems $1, 2, \dots, n$ hold. Thus, for the generalized case of these theorems, we have Theorem A.

Theorem A. Consider the set where G is acting on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Then, for the isotropy group G_x, if $|G_x| = n!$, there exists an open set $U \subseteq \mathbb{R}^n$ such that $x \in U$ yields the restriction of π defined by

$$\pi_U : U \to \pi(U).$$

In particular, there is an open set $(\pi(U))$ containing $\pi(x)$ that is homeomorphic to an open subset of the half-space $H = \{(x_1, x_2, \dots, x_n) | x_1 \geq x_2 \geq \cdots \geq x_n\}$, i.e. the function ρ defined by

$$\rho_U : \pi(U) \to H$$

is a homeomorphism. Furthermore, this shows that $(\pi(U))$ is homeomorphic to $\mathbb{R}^n/G_x$, that is, that α_U defined by

$$\alpha_U : \pi(U) \to \alpha(U)$$

is a homeomorphism.

Note that whenever $|G_x| = n!$, we have $x = (x_1, x_2, \dots, x_n)$ where $x_1 = x_2 = \cdots = x_n$.

Now, with all of the information we have learned about our chord space set, we now have almost all of the material needed to prove that this space is in fact an orbifold. First though, we need two more definitions, that is, we need the definition for a Hausdorff space and, of course, the definition for an orbifold.

Definition 6. *A topological space X is called a* ***Hausdorff space*** *if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint* (Munkres, p. 98).

Definition 7. *An* ***orbifold*** *O is a space locally modeled on $\mathbb{R}^n$ modulo finite group actions. In other words, O consists of a Hausdorff space X_O, with some additional structure: X_O is to have a covering by a collection of open sets $\{U_i\}$ closed under finite intersections. To each U_i is associated a finite group Γ_i, an action of Γ_i on an open subset $\widetilde{U}_\iota$ of $\mathbb{R}^n$ and a homeomorphism $\varphi_i: U_i \approx \widetilde{U}_i/\Gamma_i$.*

Whenever $U_i \subset U_j$, there is to be an injective homomorphism

$$f_{ij}: \Gamma_i \hookrightarrow \Gamma_j$$

and an embedding

$$\tilde{\varphi}_{ij}: \widetilde{U}_i \hookrightarrow \widetilde{U}_j$$

equivariant with respect to f_{ij} (i.e., for $\gamma \in \Gamma_i, \tilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\varphi}_{ij}(x)$) such that the diagram below commutes.

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\varphi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i"] & \tilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\varphi_i"] \arrow[r, phantom, "\subset"] & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}$$

We regard $\tilde{\varphi}_{ij}$ as being defined only up to composition with elements of Γ_j, and f_{ij} as being defined up to conjugation by elements of Γ_j. It is not generally true that $\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ when $U_i \subset U_j \subset U_k$, but there should exist an element $\gamma \in \Gamma_k$ such that $\gamma\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ and $\gamma \cdot f_{ik}(g) \cdot \gamma^{-1} = f_{jk} \circ f_{ij}(g)$.

(Thurston, p. 300-301)

In short, this definition says that an orbifold is a space that, when looked at up close, functions like $\mathbb{R}^n$. With the assistance of some of our previously stated, we will now prove that $\mathbb{R}^2/G$ is an orbifold, which will thus give us the tools needed to prove the generalized case $\mathbb{R}^n/G$. Note that here Γ_j and Γ_i will be the isotropy groups G_x and G_y, respectively.

Proof.
Consider the set $\mathbb{R}^2/G$.
There are six cases of relationships that need to be explored to prove that this is an orbifold, which can been visualized through Figure 6-3 and Figure 6-4:

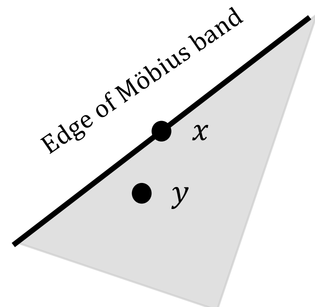


Figure 6-3

Case 1 - When $\tilde{U}_{\tilde{y}} \subseteq \tilde{U}_{\tilde{x}}$, $|G_x| = |G_y|$, and thus γ is the identity.
Case 2 - When $\tilde{U}_{\tilde{y}} \subseteq \tilde{U}_{\tilde{x}}$, $|G_x| \neq |G_y|$, and thus γ is the identity.
Case 3 - When $\tilde{U}_{\tilde{y}} \nsubseteq \tilde{U}_{\tilde{x}}$, $|G_x| = |G_y|$, and thus γ is not the identity.
Case 4 - When $\tilde{U}_{\tilde{y}} \nsubseteq \tilde{U}_{\tilde{x}}$, $|G_x| \neq |G_y|$, and thus γ is not the identity.
Case 5 - When $\tilde{U}_{\tilde{y}} \nsubseteq \tilde{U}_{\tilde{x}}$ and $|G_x| \neq |G_y|$, but $\tilde{U}_{\tilde{y}} \cap \tilde{U}_{\tilde{x}} \neq \emptyset$, and thus γ is some form of the identity.
Case 6 - When $\tilde{U}_{\tilde{y}} \nsubseteq \tilde{U}_{\tilde{x}}$ and $|G_x| = |G_y|$, but $\tilde{U}_{\tilde{y}} \cap \tilde{U}_{\tilde{x}} \neq \emptyset$, and thus γ is some form of the identity.

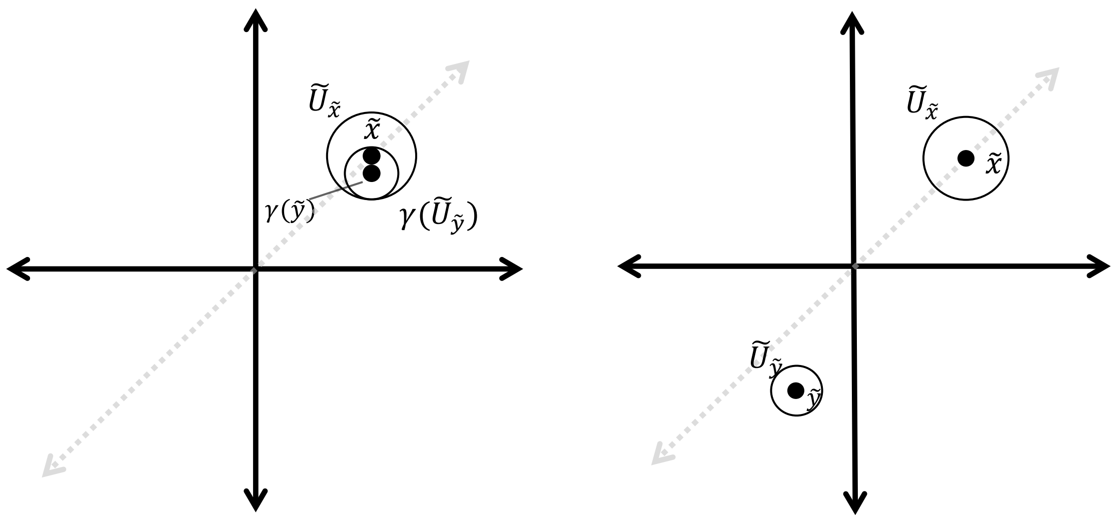


Figure 6-4

Note that Cases 1 and 3 each have two cases within them: when $|G_x| = 1$ and when $|G_x| = 2$, while in Cases 2 and 4, since $|G_x| \neq |G_y|$ these cases are both satisfied. Furthermore, because we already proved that $(\pi(U))$ is homeomorphic to $\mathbb{R}^2/G_x$ for both $|G_x| = 1$ and $|G_x| = 2$, Cases 1 and 2 clearly satisfy the needed relationships; since we have $\gamma = \tilde{\varphi}_{\tilde{y}\tilde{x}} = id$ we have the following relations:

- $\tilde{\varphi}_{\tilde{y}\tilde{x}}/\Gamma_{\tilde{y}} = \tilde{\varphi}_{\tilde{y}\tilde{x}} = id$
- $\varphi_{\tilde{y}} = \varphi_{\tilde{x}}$
- $f_{\tilde{y}\tilde{x}} = id$

And thus Cases 1 and 2 clearly satisfy the relations suggesting this space is an orbifold.
Additionally, the proof for Cases 5 satisfies the proof for Case 6, yet this proof not going to be provided here; for our purposes we will assume both these cases to be true.
Thus, to finish proof, we must show that these relationships hold in Cases 3 and 4. The proof for these two cases is the same, and is as follows:
Consider the relationships in the following diagram on p. 41, which reflects the diagram from Definition 5 under the condition of Case 5 in our space $\mathbb{R}^2/G$:

$$
\begin{tikzcd}
\widetilde{U}_{\tilde{y}} \arrow[r, "\tilde{\varphi}_{\tilde{y}\tilde{x}}"'] \arrow[d] & \widetilde{U}_{\tilde{x}} \arrow[d] \\
\widetilde{U}_{\tilde{y}} \arrow[r, "\tilde{\varphi}_{\tilde{y}\tilde{x}}"'] & \widetilde{U}_{\tilde{x}} \arrow[d, "f_{\tilde{y}\tilde{x}}"] \\
 & \widetilde{U}_{\tilde{x}}/S_2 \\
U_y \arrow[uu, "\varphi_{\tilde{y}}"] \arrow[r, phantom, "\subset"] & U_x \arrow[u, "\varphi_{\tilde{x}}"']
\end{tikzcd}
$$

Here, since $\Gamma_{\tilde{y}} = G_y = I_y = \{id.\}$ we have $\widetilde{U}_{\tilde{y}}/\Gamma_{\tilde{y}}$=$\widetilde{U}_{\tilde{y}}$ and $\widetilde{U}_{\tilde{x}}/\Gamma_{\tilde{y}}$=$\widetilde{U}_{\tilde{x}}$.
Somewhat similarly, since $\Gamma_{\tilde{x}} = G_x = I_x = S_2$, we have $\widetilde{U}_{\tilde{x}}/\Gamma_{\tilde{x}}$=$\widetilde{U}_{\tilde{x}}/S_2$.
Now, clearly, since $\widetilde{U}_{\tilde{y}} = \widetilde{U}_{\tilde{y}}/\Gamma_{\tilde{y}}$ and $\widetilde{U}_{\tilde{x}} = \widetilde{U}_{\tilde{x}}/\Gamma_{\tilde{x}}$, the identity is a homeomorphism exists between the pairs $\widetilde{U}_{\tilde{y}}$ and $\widetilde{U}_{\tilde{y}}/\Gamma_{\tilde{y}}$and $\widetilde{U}_{\tilde{x}}$ and $\widetilde{U}_{\tilde{x}}/\Gamma_{\tilde{x}}$.
Furthermore, since by our assumption $\widetilde{U}_{\tilde{y}} \not\subseteq \widetilde{U}_{\tilde{x}}$ but $\gamma(\widetilde{U}_{\tilde{y}}) \subseteq \widetilde{U}_{\tilde{x}}$, we have the embedding $\tilde{\varphi}_{\tilde{y}\tilde{x}} = \gamma$.
Finally, Theorems 1 and 2 prove that both $\varphi_{\tilde{x}}$ and $\varphi_{\tilde{y}}$ are homeomorphisms.
Thus, we must only show that there exists a relation $f_{\tilde{y}\tilde{x}}$ to show that this space is an orbifold.
However, there clearly exists a projection between $\widetilde{U}_{\tilde{x}}$ and $\widetilde{U}_{\tilde{x}}/S_2$.
This thus satisfies as our relation.
Therefore, this proves that $\mathbb{R}^2/G$, the space in which two-note chords exist, is an orbifold. □

The proof for the general case, $\mathbb{R}^n/G$, is similar, however it involves the consideration of many more cases and induction, and for that reason it is not included here. However, because of the relationship between the spaces of two, three, and four-note chords, it is clear that each of these spaces should follow patterns in which the relationships needed to satisfy the proof that the space is an orbifold should hold. Thus we conclude that general chord space is an orbifold.

Based on how orbifolds function, we can make some predictions on some properties of our maps to come in the next two chapters. First, because this chord space, when looked at closely, resembles $\mathbb{R}^n$, for the most part inverting the maps shouldn't affect the line and contour of the resulting piece. Alternatively however, since orbifolds have unique properties at the edges of the space, whenever the inversion is close to the edges of our space will be when the line and contour of our resulting piece will be affected. Thus, looking out for these characteristics we continue on to the next chapter where we will begin the topological mapping of our original pieces.

7 Analysis and Transcription of Mozart's *Ave Verum Corpus*

In this piece, there are two sets of voices that are functioning in pairs: Soprano/Alto and Tenor/Bass. This is most apparent when viewing the lyrical lines of the Soprano and Alto parts, which move together, as do the Tenor and Bass. For example, in mm. 3-4, the Soprano and Alto parts move in quarter notes with matching appoggiaturas and suspensions, while the Tenor and Bass lines remain in half notes. Additionally, in m. 25, the Soprano and Alto lines enter together, while the Tenor and Bass parts wait to enter a measure later. For this reason, this piece will be analyzed with two graphs, each in two-note chord space.

This topological analysis will be done phrase-by-phrase, where the maps of the transitions between phrases will be determined similarly but separately from the phrases themselves. This piece has eight total phrases with a short codetta at the end. The following nine maps in Figure 7-1 and 7-2 represent the topological maps of these nine phrases for the Soprano and Alto parts.

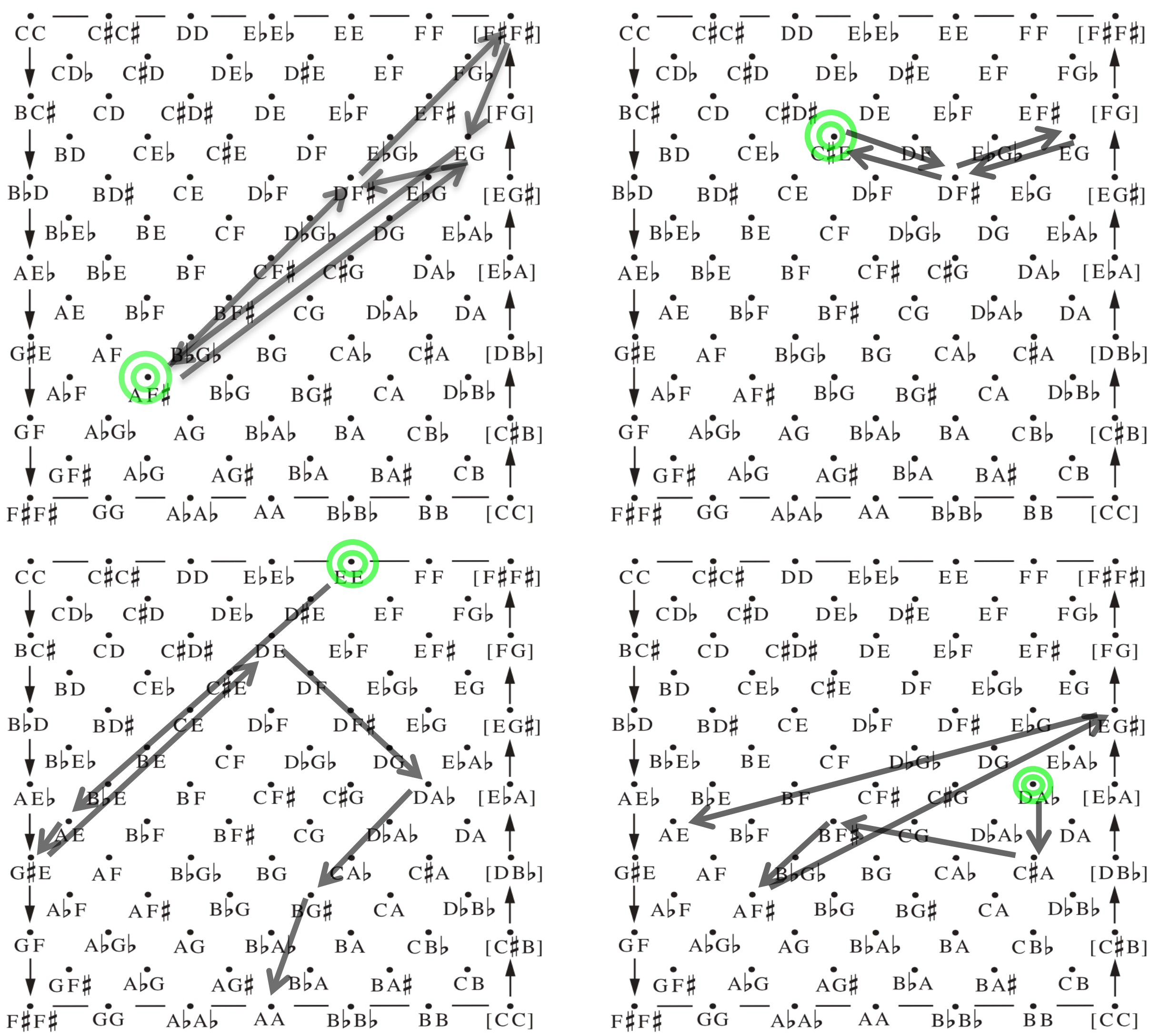


Figure 7-1

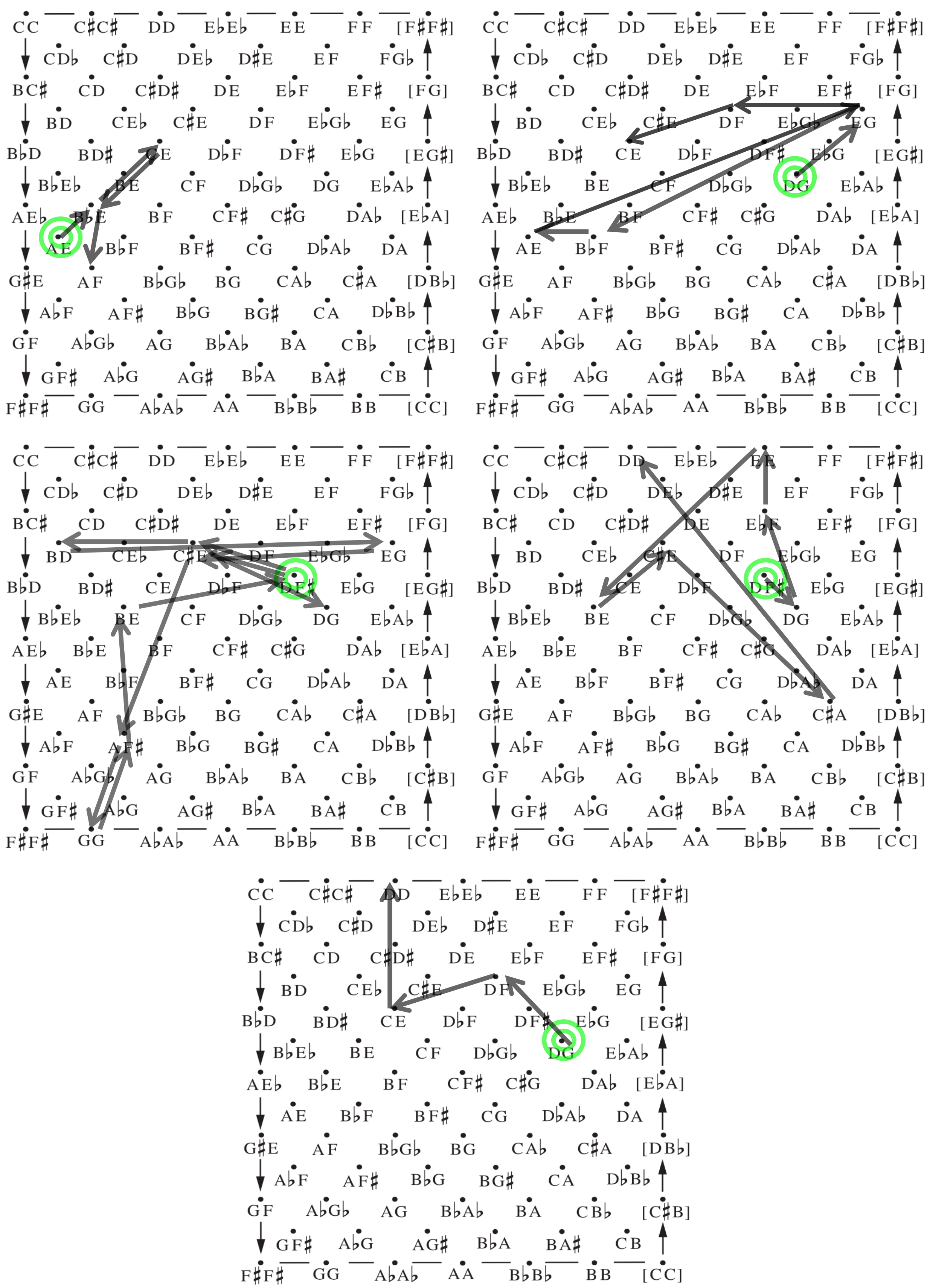

CC C♯C♯ DD E♭E♭ EE FF [F♯F♯]
CD♭ C♯D DE♭ D♯E EF FG♭
BC♯ CD C♯D♯ DE E♭F EF♯ [FG]
BD CE♭ C♯E DF E♭G♭ EG
B♭D BD♯ CE D♭F DF♯ E♭G [EG♯]
B♭E♭ BE CF D♭G♭ DG E♭A♭
AE♭ B♭E BF CF♯ C♯G DA♭ [E♭A]
AE B♭F BF♯ CG D♭A♭ DA
G♯E AF B♭G♭ BG CA♭ C♯A [DB♭]
A♭F AF♯ B♭G BG♯ CA D♭B♭
GF A♭G♭ AG B♭A♭ BA CB♭ [C♯B]
GF♯ A♭G AG♯ B♭A BA♯ CB
F♯F♯ GG A♭A♭ AA B♭B♭ BB [CC]

Figure 7-2

Using these nine maps, we will create our new map for the SA parts of our transcribed piece through inversion of the paths. To minimize error, we will begin on the same starting chord as that in Mozart's motet. We will begin by simply inverting our first map. Starting on AF#, we take our first original movement to DF# and invert this path as seen below in orange [A]. Note that if we hit the top or bottom of our space that we reflect off of it, and if we hit either of the sides we continue on that pair of pitches on the opposite side of the space going in the opposite direction.

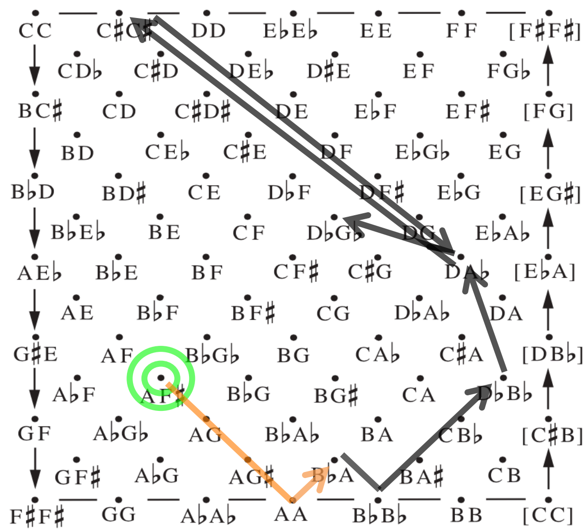


Figure 7-3

The rest of the movements made in this phrase are shown in black and are created according to the rules of our space and inversion of the previous paths. Hence the SA parts of our first phrase end on D-flat and G-flat. Now, in order to get our starting pitches for phrase two we must look at the path from the ending pitches of the original phrase one and the beginning pitches of the original phrase two. Phrase one ends on DF# and phrase two begins on C#E. Inverting this path and moving it to the ending pitches of our new phrase one we get starting pitches new phrase two of BF.

Using similar methods, the new maps for the last eight phrases of the SA lines are shown below in Figure 7-4 and 7-5.

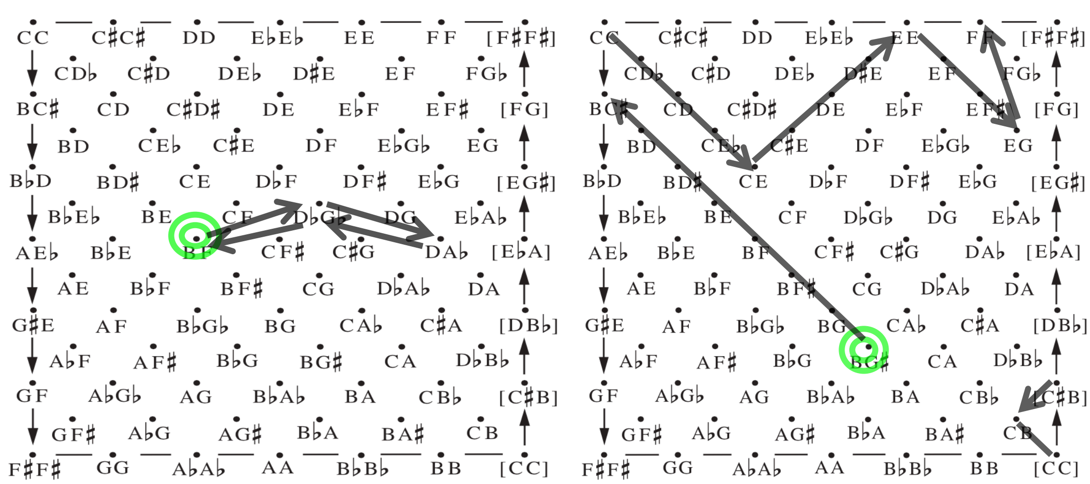


Figure 7-4

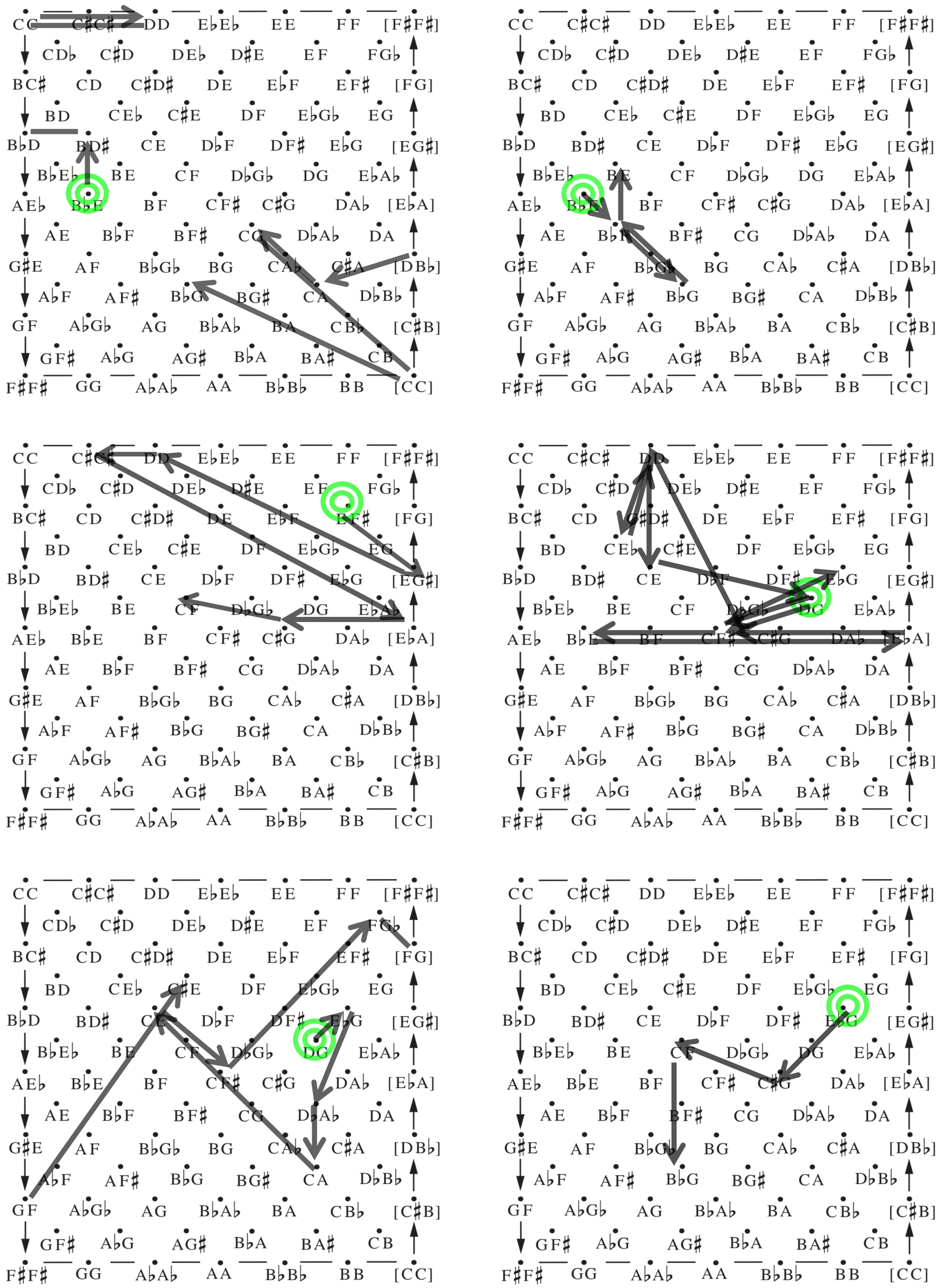


Figure 7-5

Thus, it is clear from this first set of mappings in the two-note chord space that inversion functions normally until it comes into contact with an edge, which is consistent with the behavior of neighborhoods on the edges of orbifolds as discussed in Chapter 6. We will now map and apply the same functions to the Tenor and Bass parts. The original maps are shown next in Figure 7-6.

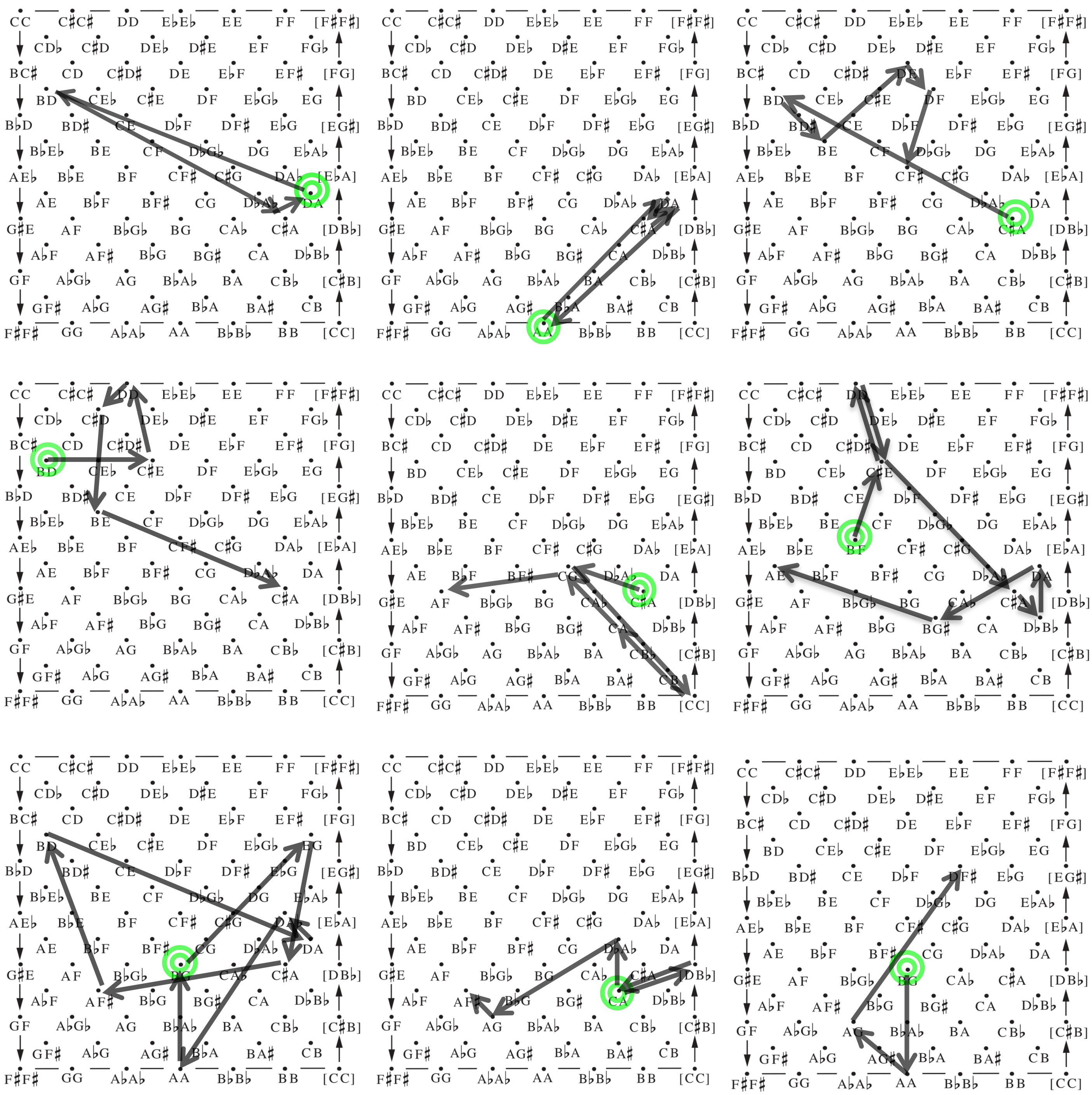


Figure 7-6

Now, with these graphs, we will repeat our process [A] from earlier to get the maps for the new piece. Here the process will be completed and not explained again. These new maps are shown below in Figure 7-7.

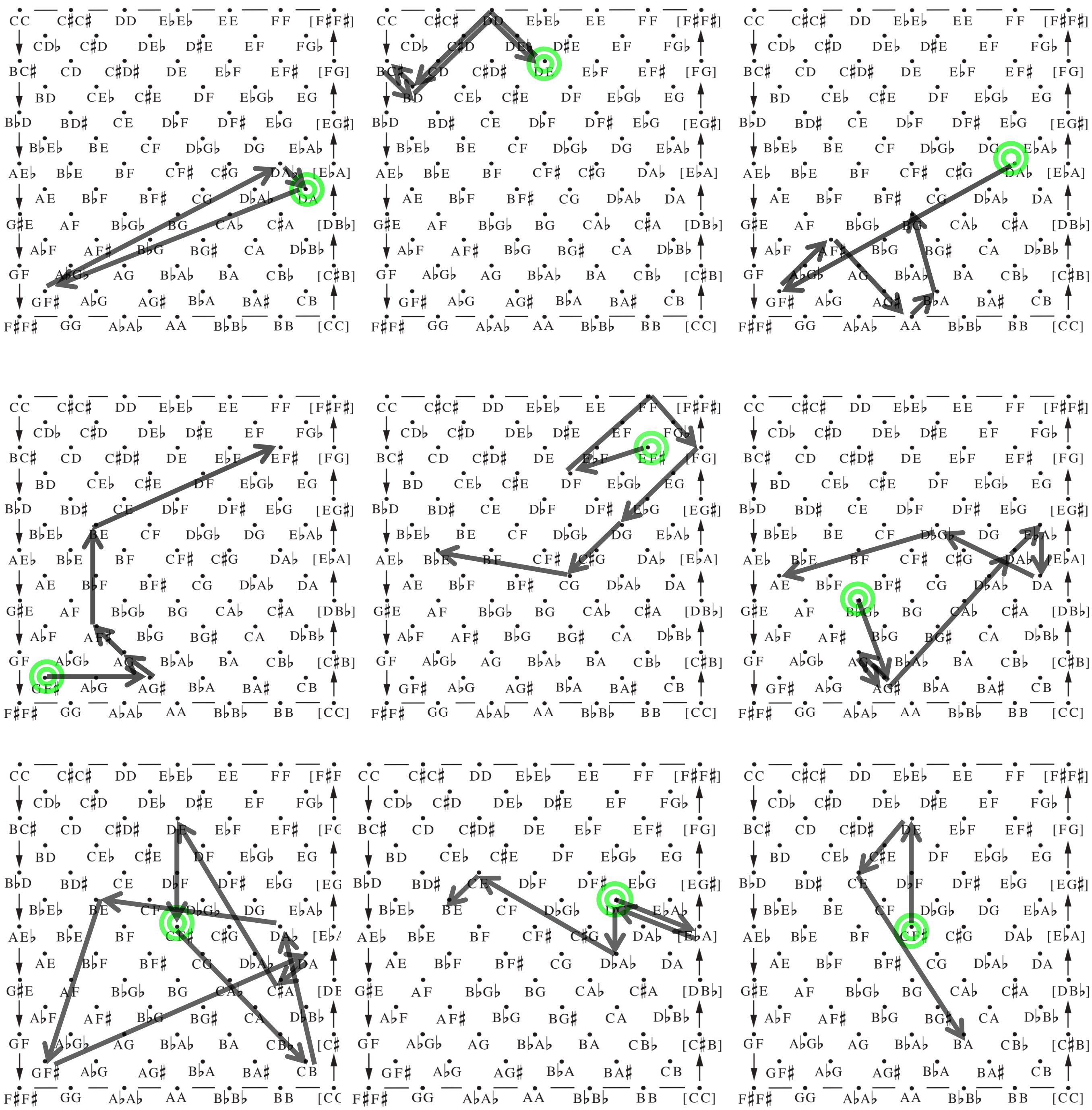


Figure 7-7

Using these, a new piece will now be transcribed. Because we are working with pitch progressions and not note durations, we will use Mozart's piece as a guide as to when to perform each progression. Furthermore, we will use our better judgment to determine which voices move to which pitches based on what pitch they are moving from and where it lies in the clef. Additionally, there will be breath marks written to signify phrase breaks. The result of our transcription is the piece shown on pp. 77. We have now composed our first piece using topological mapping.

8 Analysis and Transcription of Liszt's *Ave Verum*

Now we will topologically invert Liszt's *Ave Verum* using the method outlined at the end of Chapter 4. Again, because there's no direct way to look at a map in four-dimensional space, instead we will be looking at each harmonic progression and invert each pitch's movement individually. Because we have already completed an example of this, in Figures 8-1 to 8-7 below there are two columns: a chart of the harmonic progressions in Liszt's *Ave Verum* and, next to it, the results of this inversion which give us our new piece. Each is grouped into phrases, of which there are ten.

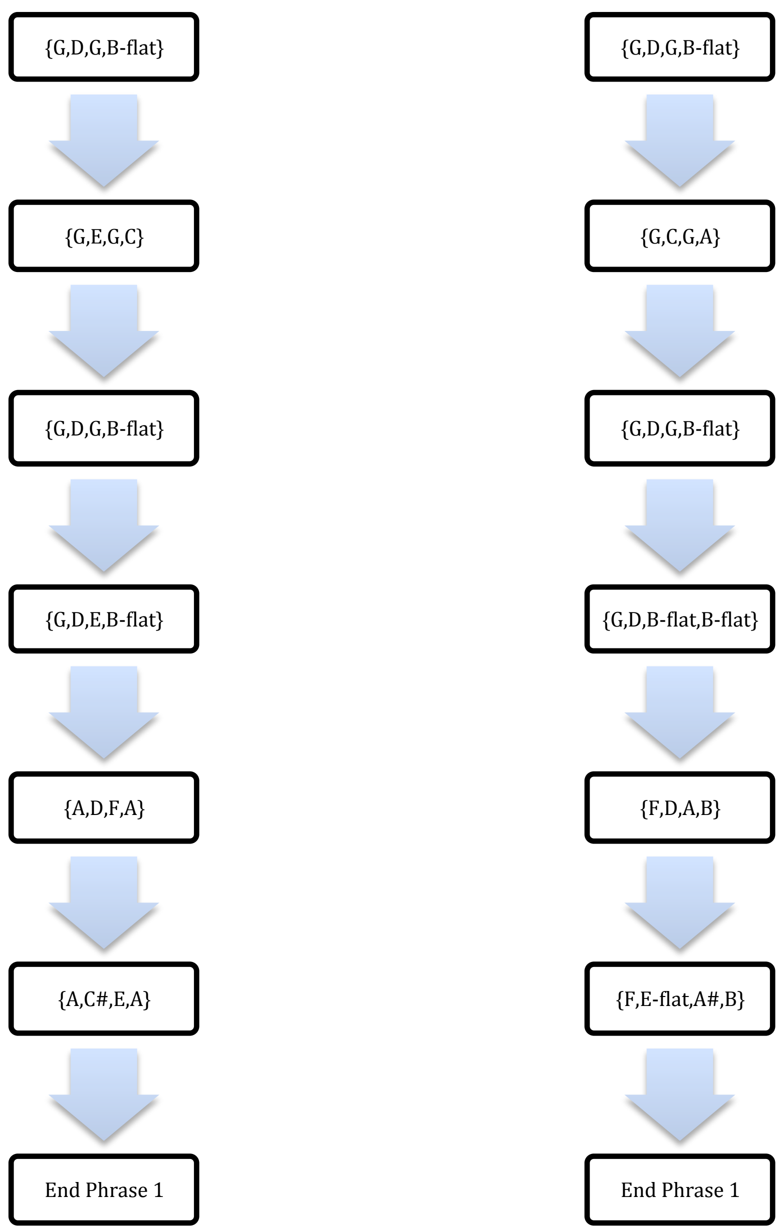


Figure 8-1

Note that transitions between phrases will be handled similarly to those in Chapter 7, that is, the progression in the first piece between the last chord of the first phrase and the first chord of the new phrase will be inverted to find the first chord of the new second phrase. This will keep the overall inversion of the piece consistent.

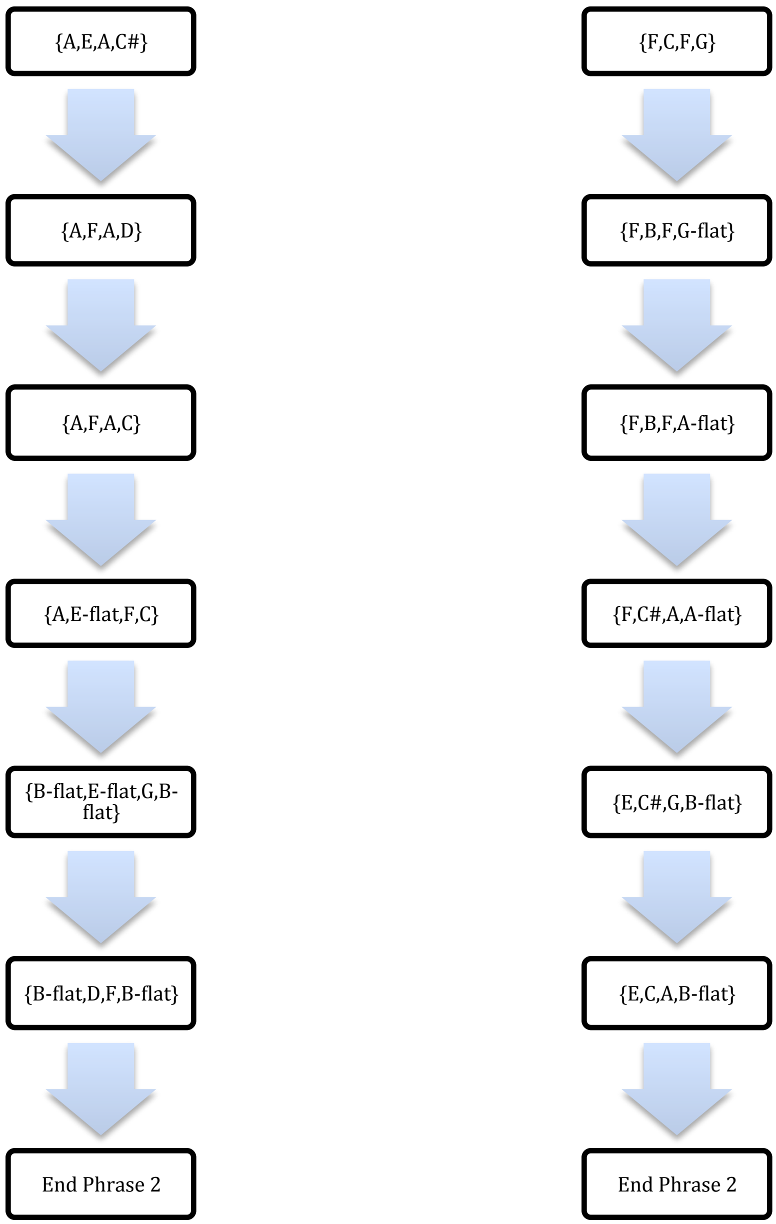


Figure 8-2

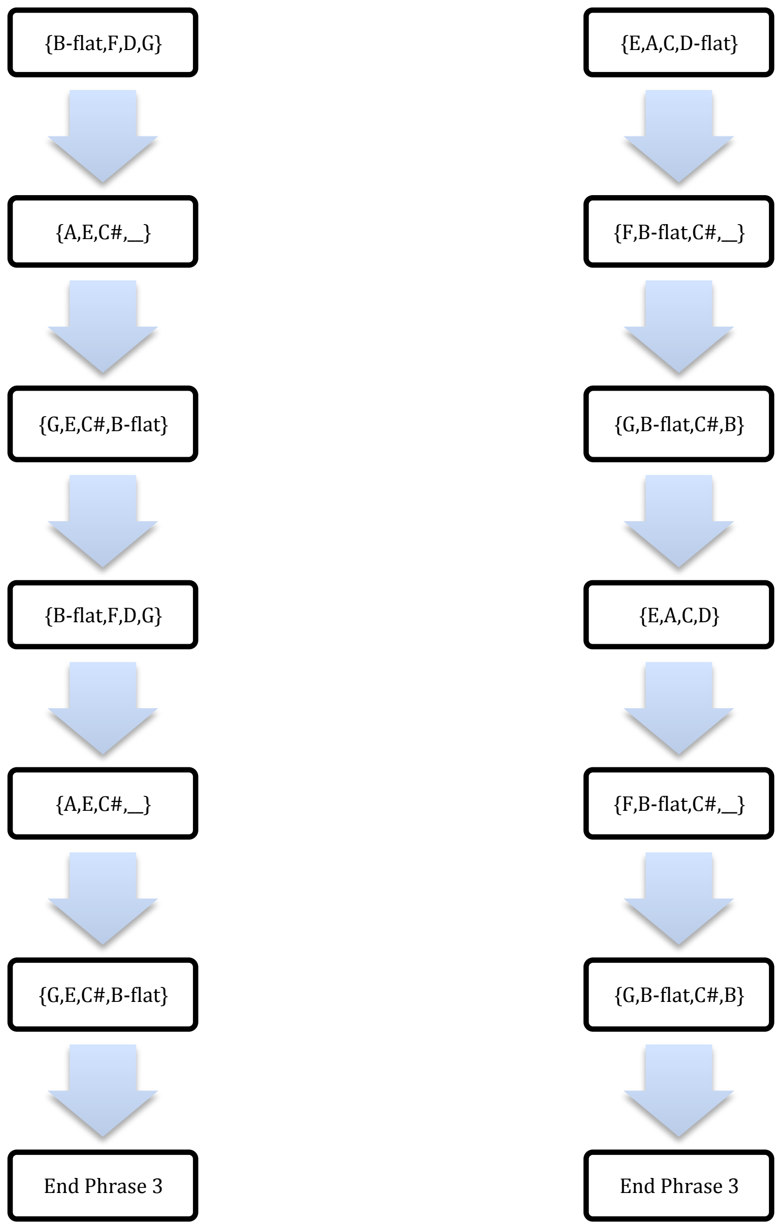


Figure 8-3

Note that in phrase three, in Figure 8-3, the second and fifth chords move into three-note chord space. Here, we will skip these chords when finding the pitches for the Bass voice.

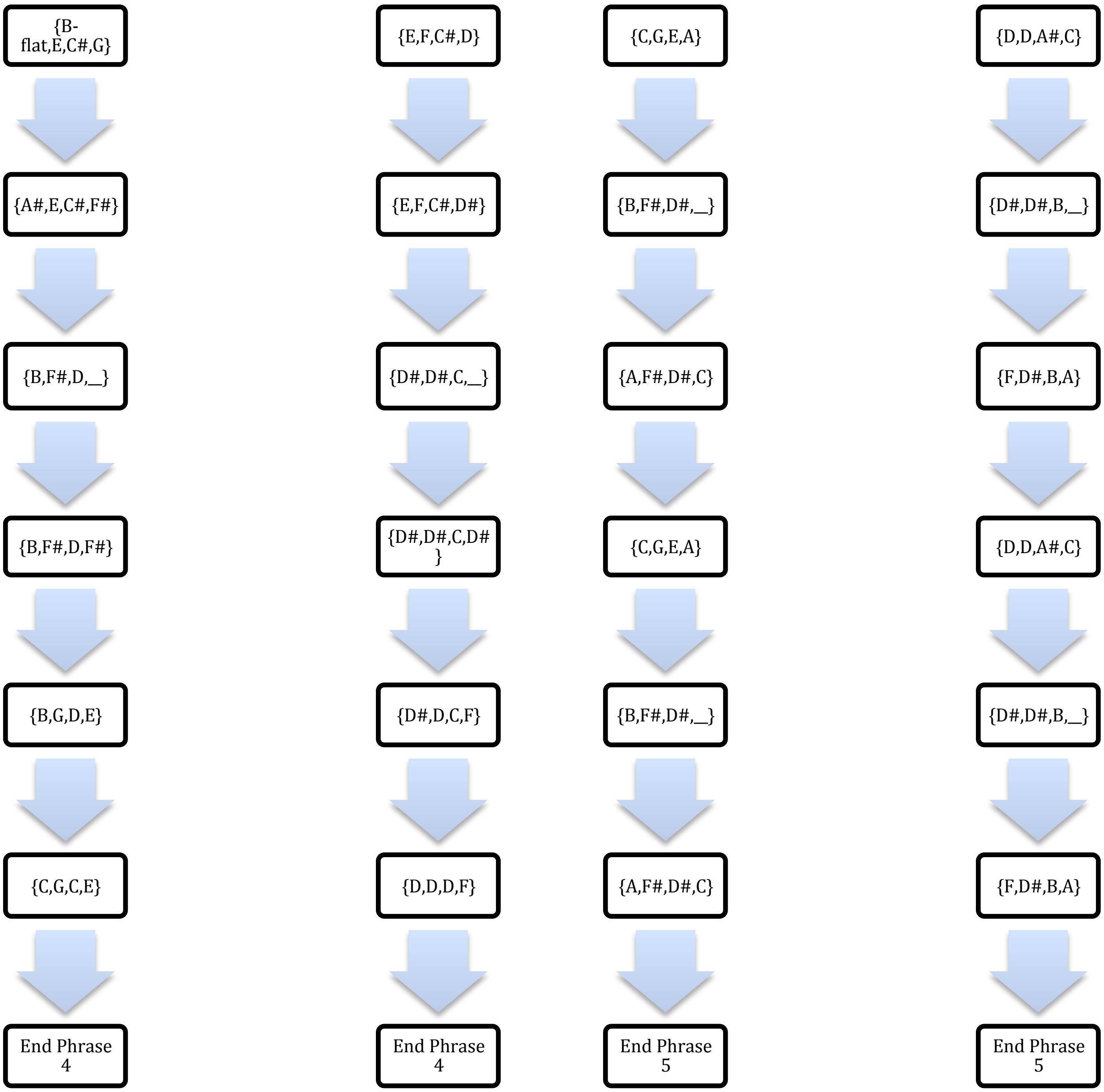

{B-flat,E,C#,G}
{A#,E,C#,F#}
{B,F#,D,_}
{B,F#,D,F#}
{B,G,D,E}
{C,G,C,E}
End Phrase 4
{E,F,C#,D}
{E,F,C#,D#}
{D#,D#,C,_}
{D#,D#,C,D#}
{D#,D,C,F}
{D,D,D,F}
End Phrase 4
{C,G,E,A}
{B,F#,D#,_}
{A,F#,D#,C}
{C,G,E,A}
{B,F#,D#,_}
{A,F#,D#,C}
End Phrase 5
{D,D,A#,C}
{D#,D#,B,_}
{F,D#,B,A}
{D,D,A#,C}
{D#,D#,B,_}
{F,D#,B,A}
End Phrase 5

Figure 8-4

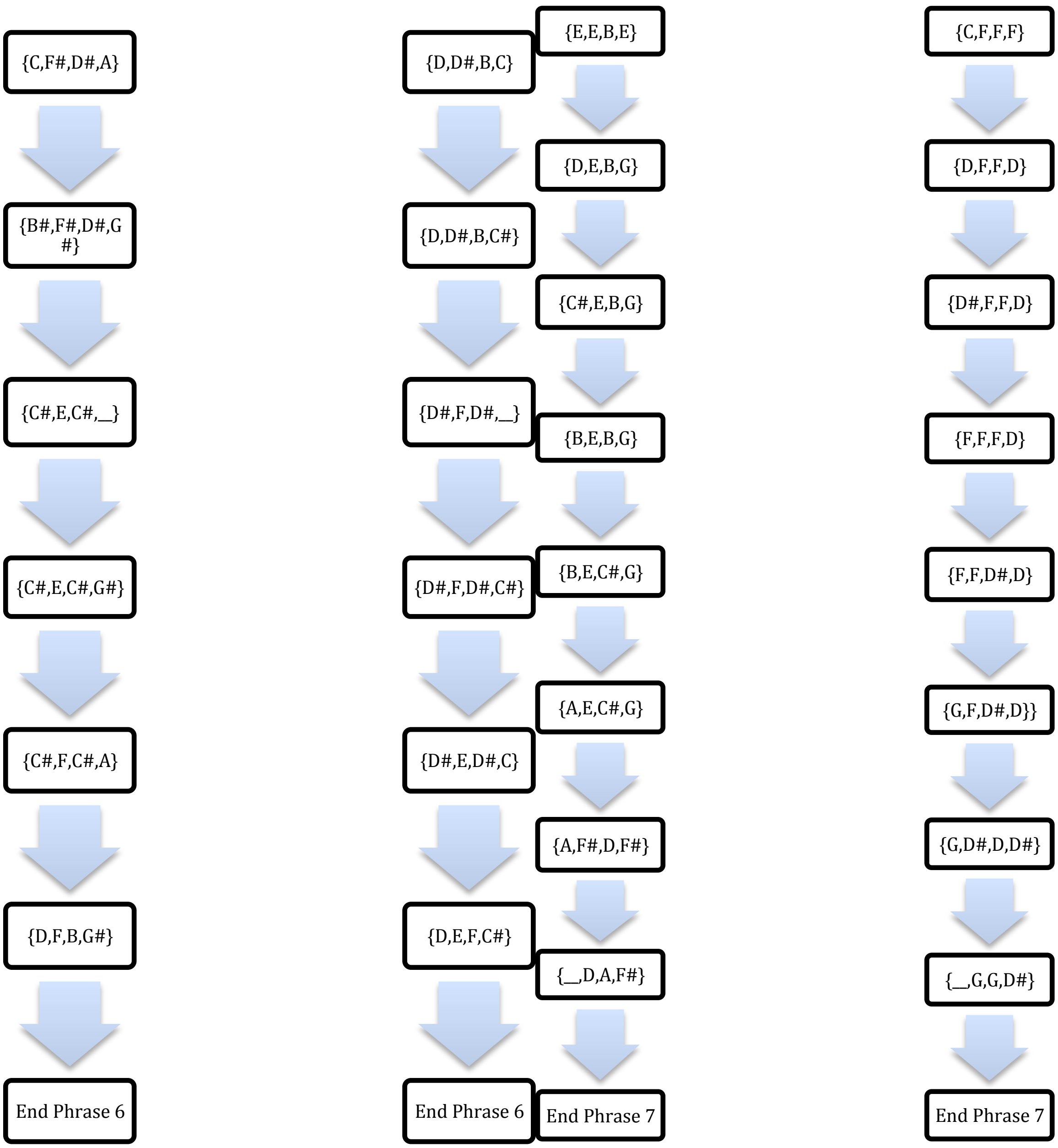
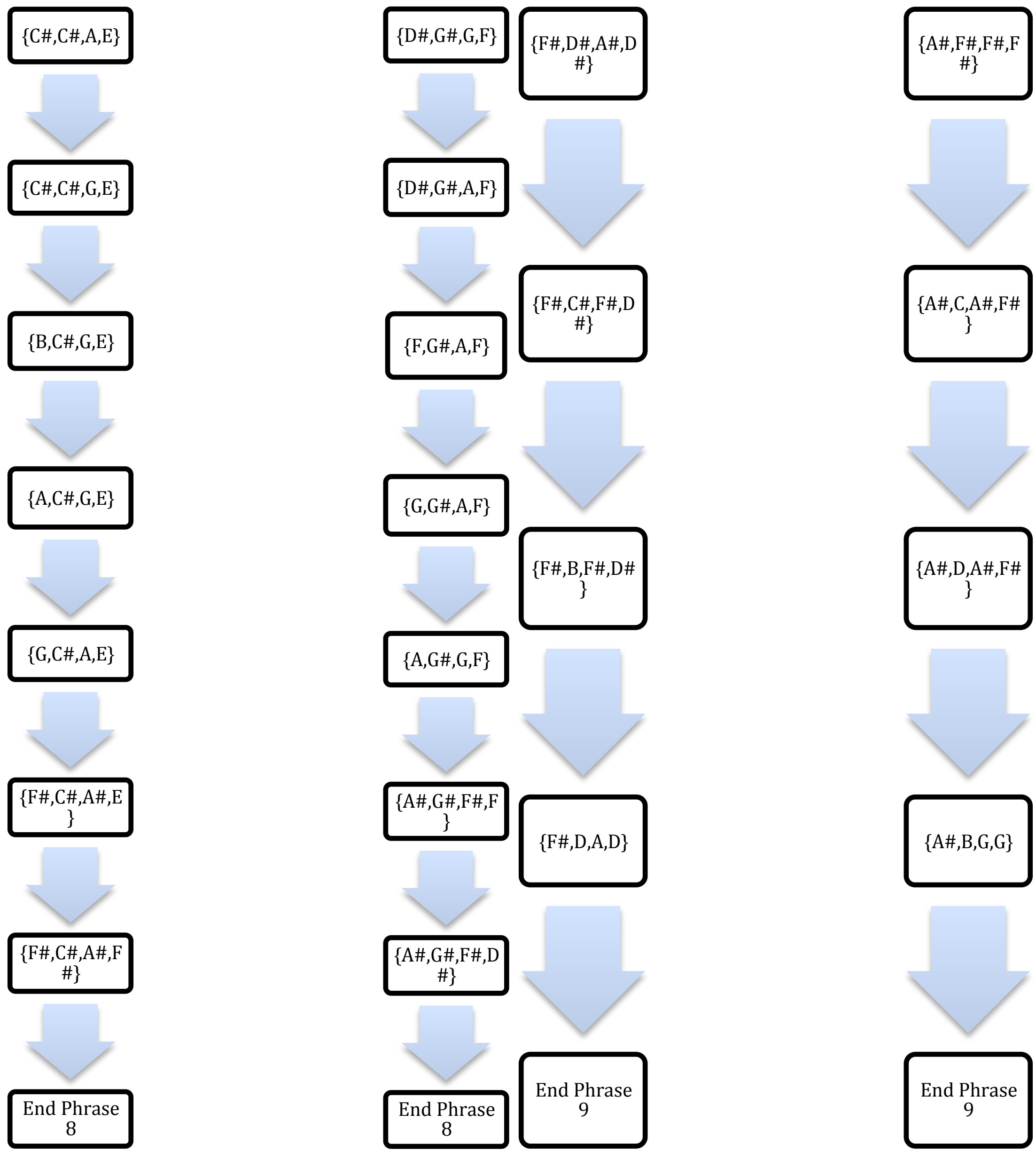

{C,F#,D#,A}
{B#,F#,D#,G#}
{C#,E,C#,__}
{C#,E,C#,G#}
{C#,F,C#,A}
{D,F,B,G#}
End Phrase 6
{D,D#,B,C}
{D,D#,B,C#}
{D#,F,D#,__}
{D#,F,D#,C#}
{D#,E,D#,C}
{D,E,F,C#}
End Phrase 6
{E,E,B,E}
{D,E,B,G}
{C#,E,B,G}
{B,E,B,G}
{B,E,C#,G}
{A,E,C#,G}
{A,F#,D,F#}
{__,D,A,F#}
End Phrase 7
{C,F,F,F}
{D,F,F,D}
{D#,F,F,D}
{F,F,F,D}
{F,F,D#,D}
{G,F,D#,D}}
{G,D#,D,D#}
{__,G,G,D#}
End Phrase 7

Figure 8-5



{C#,C#,A,E}
{C#,C#,G,E}
{B,C#,G,E}
{A,C#,G,E}
{G,C#,A,E}
{F#,C#,A#,E}
{F#,C#,A#,F#}
End Phrase 8
{D#,G#,G,F}
{D#,G#,A,F}
{F,G#,A,F}
{G,G#,A,F}
{A,G#,G,F}
{A#,G#,F#,F}
{A#,G#,F#,D#}
End Phrase 8
{F#,D#,A#,D#}
{F#,C#,F#,D#}
{F#,B,F#,D#}
{F#,D,A,D}
End Phrase 9
{A#,F#,F#,F#}
{A#,C,A#,F#}
{A#,D,A#,F#}
{A#,B,G,G}
End Phrase 9

Figure 8-6

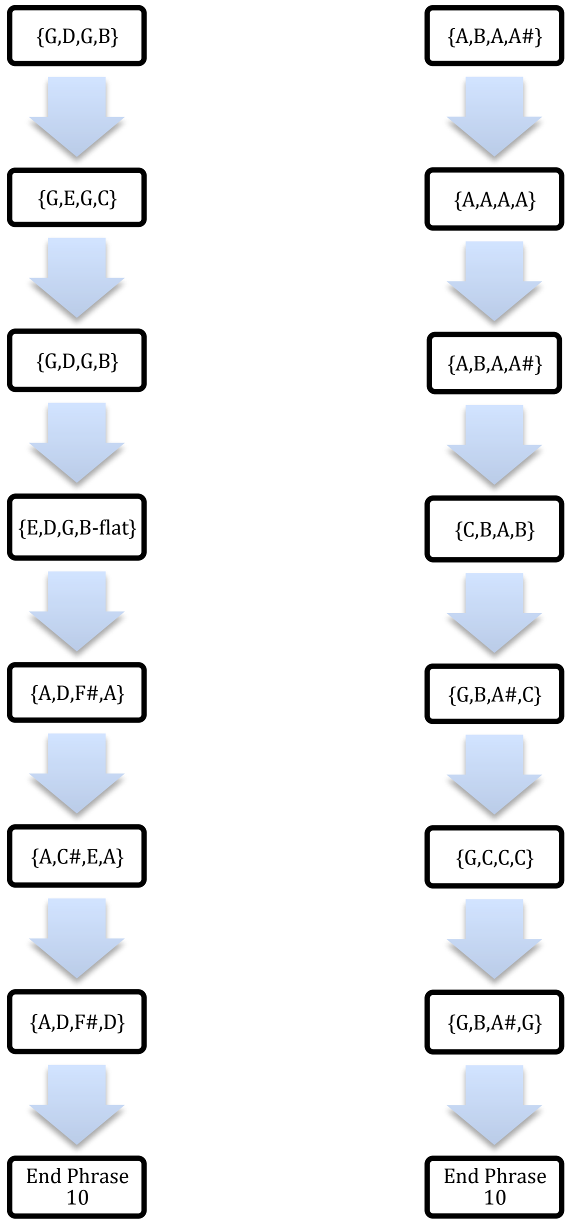


Figure 8-7

Again, using these charts, a second new piece will be transcribed using use Liszt's "Ave Verum" as a guide for when to perform each progression. The result of this second transcription is shown on pp. 80. We may now analyze these two pieces musically to determine if inverting their progressions retained any Classical or Romantic elements throughout this topological transcription.

9 Musical Analysis of Classical Period Musical Composition

Just seconds into listening to this piece one loses faith in the ability of topological mapping to reproduce harmonic structure. In short, the resulting piece is not aurally pleasing. However, as was explained in Chapter 2, the most important characteristics of Classical music are approaches and resolutions of dissonances and cadence points. Since most of the harmonic aspects of this piece are dissonant, and none of the tones were consciously composed to be "non-chord tones", we will instead use this chapter to explore the possibility of this new piece adhering to the Classical music theory rules revolving around cadence points.

There are nine "cadence points" that we will thus be looking at, each at the end of a phrase. Thus we will be looking specifically at mm. 4, 7-8, 12, 15-16, 20, 23-24, 31-32, 36, and 37-38. For each "cadence point", we will look at two things:

(1) The validity of the "cadential" chord in Classical style.
    a. If the chord is valid, does it suggest a familiar cadence?
(2) The approach to this chord.

Figure 9-1

In our first "cadence" at m. 4, shown above in Figure 9-1, the chord approached at the cadence contains the pitch class set (pcs) of {G-flat, D-flat, A, D}, which is not that of a true Classically used chord. This means that for (1), this phrase fails. Now consider the approach to this chord. The Soprano voice moves to G-flat from the A-flat above, which is movement by step. This falls well within the guidelines of the Classical style. The Alto voice approaches D-flat through D-natural, which is movement by half-step down. This again is within our guidelines, however often the use of half steps in cadences for Classical pieces would be upward movement from leading tones to tonics, which makes this less typical. Both the Tenor and Bass voices "approach" the cadence by staying in unison with the previous note, and in doing so they are working within the Classical style. Thus, our first phrase, while not pleasing to the ear, does suggest that note-to-note movement remains intact when composing topologically. This suggestion gives us hope to continue on in our analysis.

In mm. 7-8, shown below in Figure 9-2, the "cadential" chord has a pcs of {D,F,D,E} which, again, is not a traditional Classical chord due to the presence of the 9th, E. However, in other circumstances this chord would not be unusual. Consequently we again move on to criteria (2). Here, the Soprano, Alto, and Tenor voices all approach the chord by leap, more specifically by minor 3rd , major 3rd, and minor 3rd respectively. This is unusual. It is even more unusual when one sees that in this "cadence" the Bass voice approaches by a whole step upwards. This eliminates the suggestion of a leading tone and additionally goes directly against the convention of the Bass voice leaping to the cadence while the other voices move primarily by step and unison. Hence this phrase shows little promise for our purposes.

Figure 9-2

Moving on to m. 12, also shown in Figure 9-2, here we have a pcs of {F,F,B,G}, which, unlike any phrase before, can be interpreted as a G-major dominant seventh chord. While one is content that this chord is recognizable, it is unfortunately a beat too early, for seventh chords precede cadential chords and aren't cadential chords themselves in Classical music theory. However, this phrase does contain something that provides much more hope for our second criterion. Here, the Soprano voice approaches downward by step and the Bass voice approaches by leap, which, by our guidelines, are both acceptable. More noteworthy still is that both the Alto and Tenor voices approach by upward half-step, both of which suggest leading tones. However, in terms of which one is stronger, the Tenor voice features a B-flat ascending to a B-natural, which much more strongly suggests leading tone than E to F in the Alto voice due to the use of chromatic alteration. So it is seen that here there is a cadence in C-flat major or minor. This may seem insignificant, however G is the fifth of a C-chord, and so we are an augmented fifth away from having an unconventional half-cadence, which is the most promising phrase we have experienced in our topological transcription thus far.

The next cadence, in mm. 15-16, shown in Figure 9-3, has a pcs of {B-flat, G, F-sharp, E}. Due to it containing the pitch classes E, F-sharp, and G, this chord clearly fails our first criterion. Moving on to criterion two, the Soprano voice approaches by downward whole step, the Bass voice remains on a unison, and both the Alto and Tenor voices leap by a perfect fourth to our cadential chord. Despite this similarity in the interval contained in the leap, this behavior for the inner voices is unusual for our purposes. Thus, while the Soprano and Bass movements are accepted, the Alto and Tenor are breaking a rule, hence this phrase, similar to our second phrase, fails both (1) and (2).

Figure 9-3

In m. 20 above, we see that this cadence clearly fails (1) with a pcs of {B,E,B-flat,E}. Again, one might lose hope here, but as we saw in phrase three, there is potential for recognizable chords to arise throughout this composition. In terms of this phrase satisfying criterion two, the Alto voice moves downward by half-step and the Bass approaches by an upward leap of a major 3rd. Unfortunately, here the Tenor also approaches our "cadence" by an upward minor 3rd leap. However, we will ignore this blemish in our approaches because in this phrase the Soprano voice approaches our cadential chord by an ascending half-step, which, as mentioned before, suggests a leading tone. In this case, this leading tone implies a C-flat major or minor chord, which is determined to be major due to the presence of E-naturals. Now, while there isn't a clear relationship between C-flat major and the chord we have here, it does relate to the last suggested cadential key that we had: C-flat major/minor. Thus this phrase gives us hope for a new potential consistency in topological transcription: the relationship and consistency of keys between phrases.

Our next "cadence" is in mm. 23-24, shown in Figure 9-4, where our chord has a pcs of {F,C,A,E}. In contrast to the majority of our previous phrases, this chord is an F-major seventh chord, which is similar to the last recognizable chord that we found in that it is unusual for a cadential chord. Again, just the fact that the chord is recognizable is noteworthy here.

Figure 9-4

In this phrase, this chord is approached by a downward whole step in the Soprano voice, a downward half-step in the Alto and Tenor voices, and an upward whole step in the Bass voice. While all of these are acceptable by our standards, there is again the lack of a leading tone and the unusual use of half-step downward motion that disqualifies this phrase in terms of our second criterion.

With only three more phrases left in our "Classical" piece, there are some patterns of which to take note. First, there has been a recognizable chord every three phrases thus far, both of which have been seventh chords. This potentially could be connected to our method of topological transcription. Second, most phrases satisfy the second criterion, and some even suggest home keys through their movement, however when a phrase does fail the second criterion, it does so in ways critically unacceptable within the Classical style. Thus, in these last three phrases we will look to see if these patterns continue to hold true.

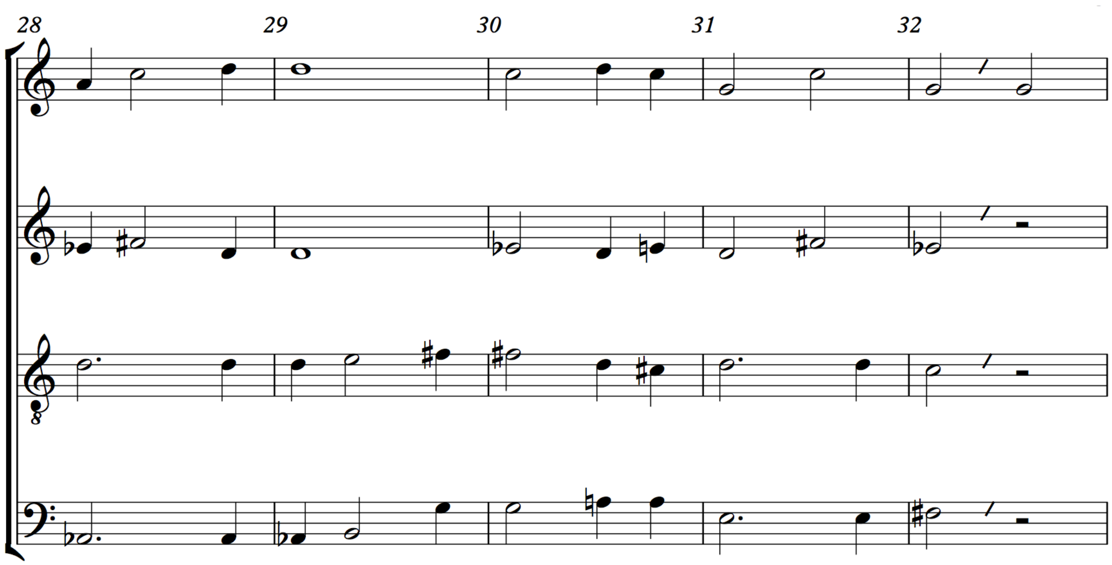


Figure 9-5

Our seventh phrase has its cadence in mm. 31-32, shown in Figure 9-5, where the cadential chord has a pcs of {G,E-flat,C,F-sharp}. At first, this chord seems to be a C minor chord, however the presence of the F-sharp, which is a tritone in the key of C minor, causes this chord to fail. One encouraging thing to notice from this is that for now our pattern seems to be holding. This chord is approached by a downward leap of a perfect fourth in the Soprano voice, a downward whole step in the Tenor voice, and an upward whole step in the Bass voice. While the Soprano voice's movement may be considered unacceptable, it is overshadowed by the approach in the Alto voice, which is by a downward augmented second. Thus, this phrase fails both (1) and (2). However, it does continue our aforementioned pattern.

The next phrase has its cadential chord in m. 36, shown in Figure 9-6, with a pcs of {E,C-sharp,B,E}. This chord suggests a C-sharp diminished seventh chord, which disrupts our pattern of chords being recognizable every three phrases, but does continue the trend of recognizable chords being seventh chords. Once again the chord in question is reasonable in Classical music theory but unusual as a cadence chord, hence we move on to (2). Here, the Soprano and Tenor voices approach by downward half-step, the Alto voice approaches by a leap of a downward perfect fourth, and the Bass voice remains on a unison. While most of these are acceptable, because we are looking for accuracy, the use of two movements downward by half-step on different pitches is unusual enough for this phrase to fail our second criterion.

Figure 9-6

Our last phrase, with a "cadence" in mm. 37-38 also shown in Figure 9-6, ends with a pcs of {B-flat,G,B,A}, which ends our piece on a very unusual and unrecognizable chord. Hence, unfortunately, our final phrase fails our first criterion. Moving on to criterion two with some hope, the Soprano voice approaches by a downward whole step, the Alto voice approaches by an upward whole step, the Tenor voice approaches by a downward half-step, and the Bass voice approaches by a downward leap of a perfect fifth. Thus, while none of the voices are moving in unacceptable intervals, we are missing a leading tone while having the unusual downward movement by half-step instead. Thus, this final phrase is an unsuccessful final phrase for our piece and indicative of the need for further research in the context of what we were trying to achieve through topological transcription.

Overall, after listening or reviewing the harmonic material of this piece, one would not be quick to think that topotonal music has a place in the future of music composition. However, there were some notable things that survived this particular topological transcription: the preservation of some acceptable movement in cadences, consistency in tonal centers when progressions were recognizable, and the presence of seventh chords as the only acceptable chords. Hence, there is some hope for topological mapping, and, as will be explored in the next chapter, perhaps it is more fitting for the transcription of Romantic style music.

10 Musical Analysis of Romantic Period Musical Composition

As in our Classical Period Composition, there are a number of things that become clear when listening to this transcription. First, while the first piece retained a recognizably active harmonic rhythm, this piece lost most of its key rhythmic elements throughout the transcription. Second, since dissonance is more acceptable in the Romantic style, the dissonances that arose here due to our use of topological transcription are less jarring. Since this piece was composed completely separate from any text, we will not be looking for any text painting here. Finally, chromaticism is common in this piece, as it was in the Classical period piece, simply due to the manner by which it was transcribed, thus that criteria will also be left out of consideration here. Thus, the key characteristics that we will look for in this chapter are:

(1) Ambiguous tonal center where there is one
(2) Modulations where there are clear keys
(3) Lack of clear cadences
(4) Use of a descending fifth progression

Before we look for these features however, first we will talk about why this piece is so much less interesting than our topologically transcribed Classical period piece. This is true mostly due to the fact that in Romantic music, and in this piece in particular, non-chord tones often provide the rhythmic and harmonic depth in a piece that makes it interesting for the listener. Thus, in taking away these tones in order to map the piece most accurately for our purposes, we stripped the piece of what made it special. Therefore, for future projects it may be suggested to re-map both pieces with the inclusion of the non-chord tones to see their effect on the topological transcription.

We will analyze this piece phrase by phrase, of which there are nine. It is also to be noted that in the analyses below it will often be necessary to omit a pitch in order to find a chord, which for our purposes will be seen as acceptable here. This will most often occur in the Bass voice due to its extensive use of non-chord tones throughout the piece.

Beginning with the first phrase from mm. 1-5, as shown in Figure 10-1, it is clear to see by the pcs of the "cadential chord", {F,E-flat, A#,B}, and that of the preceding chord, {F,D,A,B}, that this is not a true cadence. What is important for our purposes is that this is a complete lack of a cadence rather than a lack of a clear cadence. However, after analyzing all of the chords in this phrase, we see that there are three recognizable chords: G-minor, A-minor seventh, and D-minor (note that this is without the consideration of the B-natural in the Bass voice). Therefore, considering these three chords and considering what key best contains them, we conclude that this phrase is in an ambiguous G-minor. Furthermore, the roots of these chords, that is, G, D, and A are each a fifth apart, which makes reference to a circle-of-fifths progression, however is not an exact representation of such. Thus, this phrase is very promising tonally for the potential of topological mapping to hold Romantic musical characteristics.

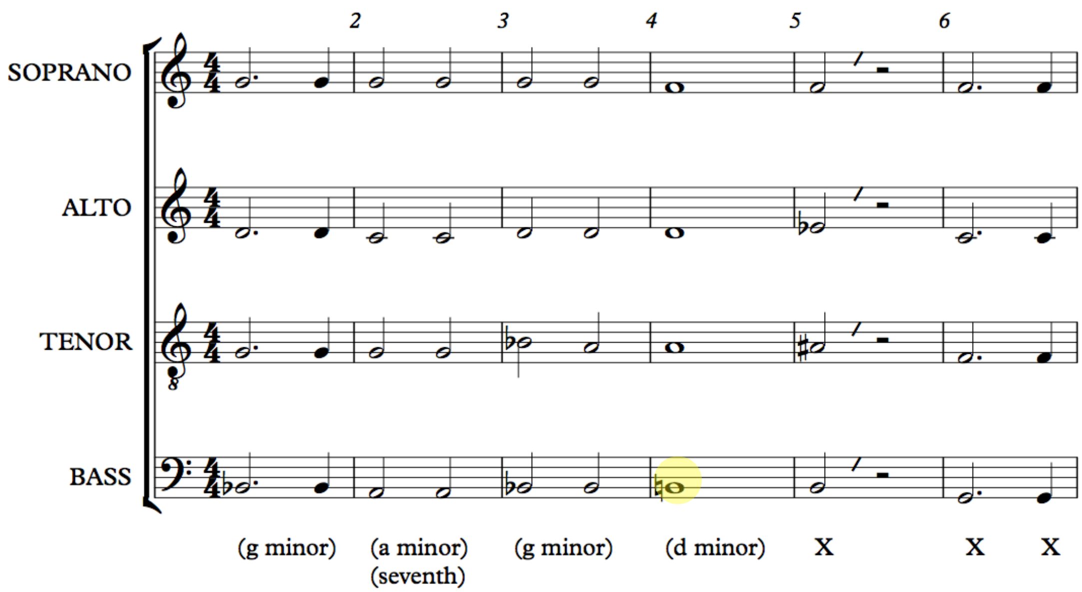


Figure 10-1

The second phrase, from mm. 6-10, has a “cadence” from a C-diminished seventh chord to an A-minor chord. This is either a i^{o7}-vi progression or a iii^{o7}-i progression, however it will be seen that the former is more likely. Since a vi chord is a common substitution for i, we also consider the chord preceding the C-diminished seventh chord, which we come to see is an augmented F chord. While this is very unusual, this progression then becomes I^{+}- v^{o7}-iii, which is vaguely reminiscent of a half cadence—just enough so for our purposes. Thus, in this phrase, we have an ambiguous but relatively valid cadence for the first time in either of our transcriptions. Alternatively however, in this phrase there are no recognizable chords except for those found in the “cadence”.

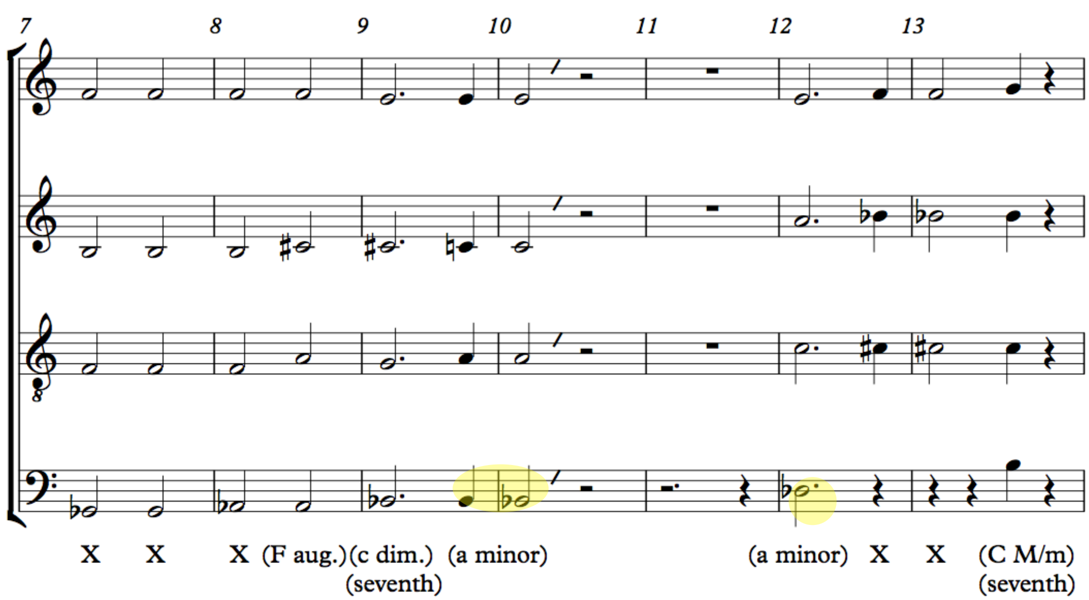


Figure 10-2

Phrase three is longer than the previous two phrases, for it is from mm. 12-19, however it doesn't contain much more material for our purposes. Recalling what this phrase was in Liszt's piece, one would remember that here there were constant suspensions, which provided the drama and harmonic intricacy of the phrase. Hence without the consideration of those, it makes sense that this section seems to be missing something. While this phrase ends clearly in the key of D-minor, the preceding two chords have pcs of {D#,D,D,F} and {D#,D#,C,D#}, both of which have no harmonic value. Thus, again here we have an outright lack of a cadence. However, as seen in the analysis of this phrase, there are three recognizable chords throughout: A-minor, C-major seventh, and D minor. These three chords thus suggest the key of D-minor, which fits well because D minor is the cadential chord. Already, it is clear that in the Romantic style there is much more hope for valid harmonic progression through topological transcription.

Figure 10-3

The fourth phrase, from mm. 21-24, resulted in a very bizarre yet unique transcription. The recognizable chords in this phrase are B-major, B-half-diminished seventh, and B-minor, all of which are related, but not in appropriate ways for harmonic progression. Thus, this phrase lacks a cadence, doesn't clearly modulate, and does not contain a circle-of-fifths progression, however it does have an ambiguous tonal center due to the variety of B-chords used. Similarly, the next phrase from mm. 25-30 fails those same three criteria, however it also lacks any recognizable chords whatsoever, and thus this phrase is clearly unacceptable. One can also recall that this phrase in Liszt's piece was a continuation of the suspension motif from the previous phrase, and thus it is not surprising that these phrases differed so much in their output due to the non-chord tones omitted in our mapping.

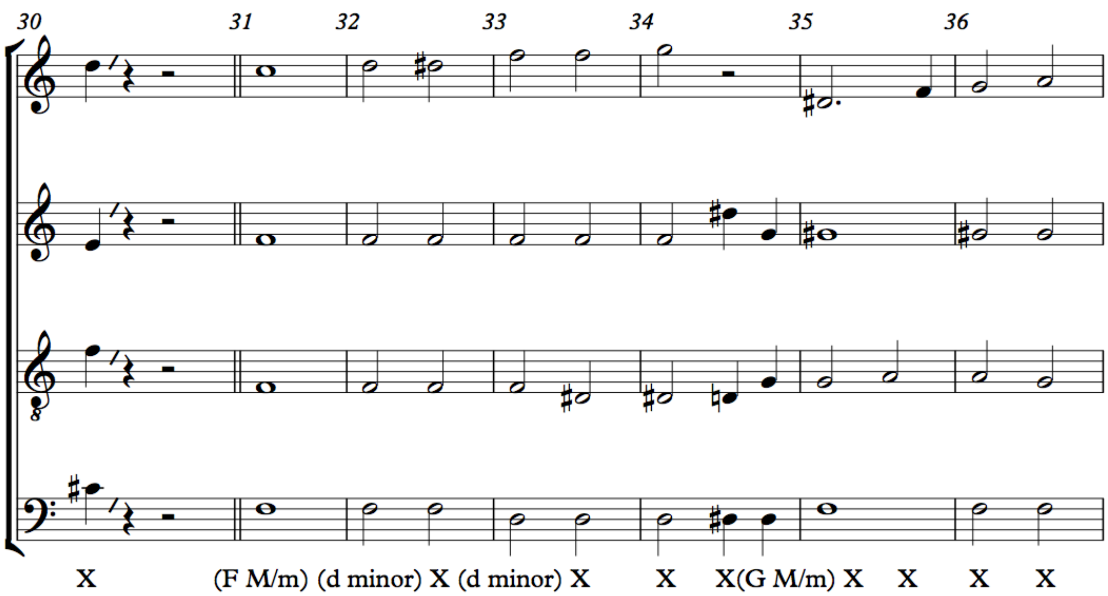


Figure 10-4

Phrase six, in mm. 31-37, begins with a few recognizable chords but quickly devolves. At the cadence point, the two chords have pcs of {A#,G#,F#,F} and {A#,G#,F#,D}, both of which don't suggest any identifiable chord. At the beginning of the phrase however there are the chords F (ambiguous of major or minor), D-minor, and G (ambiguous of major or minor), which suggest a tonal center of G. Now, only considering the five tonal centers found so far, G,F,D,B,G, it is seen that in this piece not only within phrases, but between the tonal centers there is no suggestion of a descending circle-of-fifths progression. This is with the exception of the first phrase, in which there was no suggested tonal center.

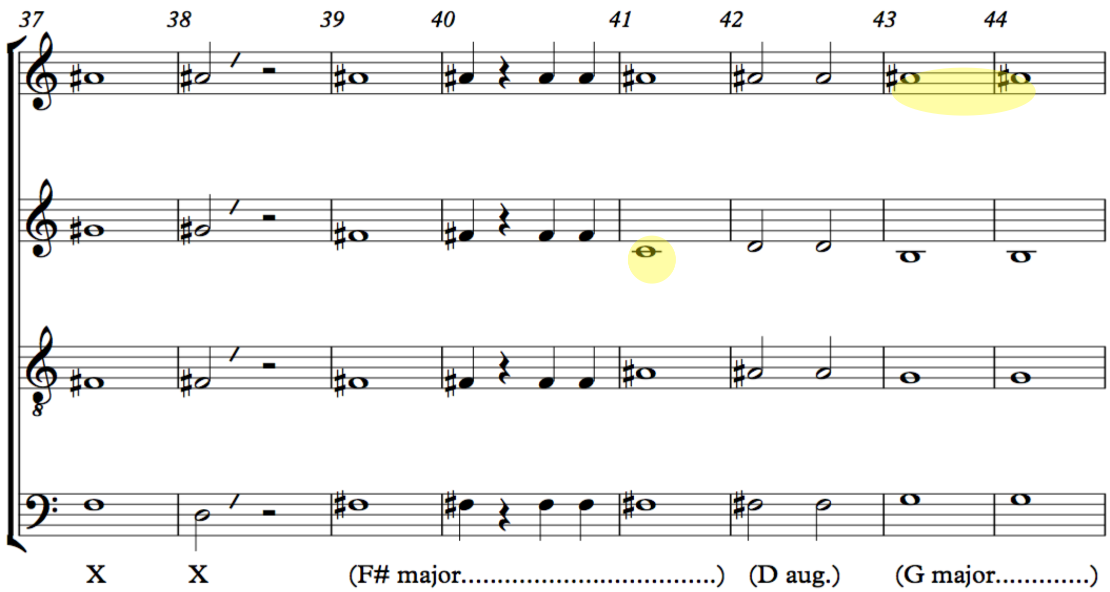


Figure 10-5

The seventh phrase from mm. 39-45 is the first phrase of the piece where there are no unrecognizable chords. This is especially surprising because in Liszt's piece this phrase contained the Neapolitan 6th and 7th chords, which are extremely unusual harmonically, even in the Romantic style. The cadential chords in this phrase are an augmented D chord and G-major, which, despite the unusual augmented chord, is similar to an authentic cadence progression in the key of G-major. Furthermore, the other recognizable chord in this phrase is a F#-major chord, which is a VII chord in this key and promising given the fact that F# is, in the key of G, the only sharp.

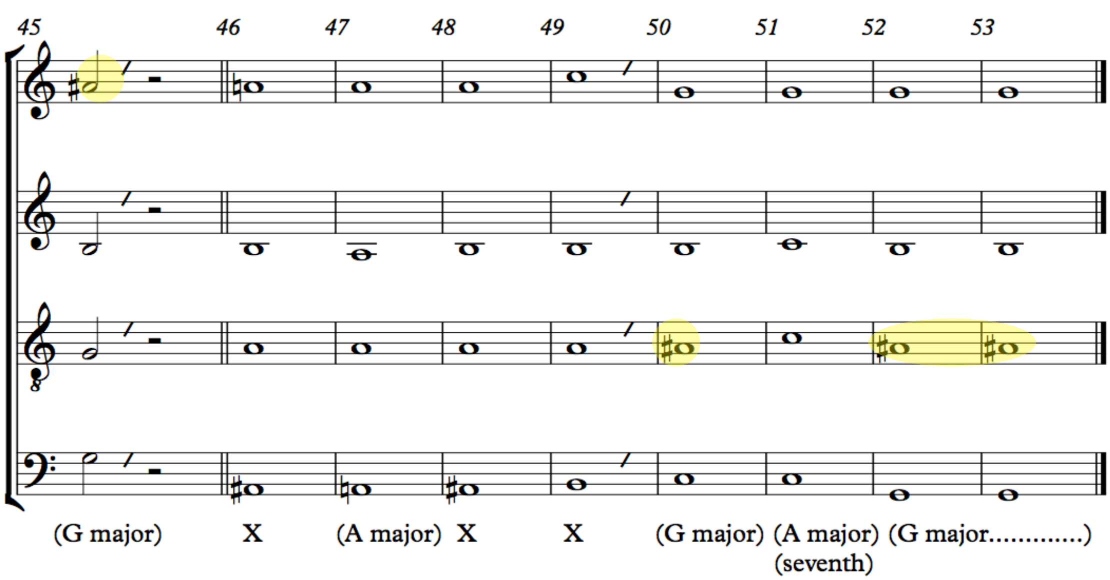


Figure 10-6

The next phrase, the second to last phrase, contains only one recognizable chord: A (ambiguous major or minor). Hence this phrase does not contain enough recognizable chords to suggest a tonal center, however this proves to be more discernable when considering the phrases surrounding it. First, the last phrase of the piece, similarly to phrase seven, contains all recognizable chords: G-major and A-minor seventh. This is an unusual progression, for the piece ends on a G-chord, hence the progression is I-ii^7-I. It is to be noted that ii^7 chords are able to function as V/V chords, and so this progression is not entirely unacceptable in this instance. Yet, while this is unusual, it remains promising given the fact that the piece ends on a progression of discernable chords. Both the previous phrase and the subsequent phrase have tonal centers of G, hence perhaps this phrase is leaning towards a tonal center of D, which is the dominant chord in the key of G and in which A would be dominant. This would make the most sense for a phrase preceding the end of a piece, however, as the composition of this piece was not aimed at aesthetic goals, here it is simply something of which to take note.

In whole, it is clear that harmonically the Romantic period piece was much more successful in its topological transcription, despite its lackluster harmonic rhythm. More interestingly, in this piece it seems that the more unconventional chords used by Liszt yielded more favorable results here, which suggests that our mapping "broke" many chords apart but brought other chords closer together. It is to be noted though that this piece featured no modulations and no clear descending fifth progressions, and thus it was not entirely successful in its accuracy in the Romantic style. Furthermore, lacking cadences instead of lacking clear cadences reduced the recognizable phrase structure of the piece, which, along with the clear structure change that came with the absence of non-chord tones, caused the piece to be very slow harmonically and, overall, boring. Therefore, while there are these clear issues with this transcription, this chapter analysis does give a more promising result for the future of topotonal music through its suggestion that some harmonic preservation is at least possible within topological transcription.

11 Conclusion

Throughout this paper, one can come to many conclusions around where chords exist in topological space, the validity of topological transcription for both Classical and Romantic period music, and in turn, what defines these periods. Most importantly however, just as Schoenberg declared the "emancipation of dissonance", this paper proves the ability of topological mapping to produce musical compositions and opens the door for the exploration of topotonal music.

First, it is clear from our proofs in Chapter 5 and Chapter 6 that chord space is an orbifold. This knowledge not only allowed us to predict the behavior of our inversion when we topologically transcribed our pieces, but it gave insight into the existing hierarchy of chords and why it exists. Perhaps further research into the behaviors of orbifolds will allow a mathematical music composer to alter their topological mappings in order to have better musical output. Thus, we have not only proven the additional connection between tonal music composition and mathematics (a relationship that is often overlooked) but we have set a foundation for further investigation of these two fields and their relationship in this way.

In terms of the successfulness of topological transcription to retain harmonic structure, we discovered two very different conclusions based on the style of piece being topologically transcribed. From this fact alone, it is worth topologically transcribing more styles of music to see the similarities and differences between each style and the results of topological transcription. Furthermore, one might thus suggest that our use of inversion is what caused the resulting similarities and differences in harmonic structure, and perhaps the use of a different alteration of the original topological map might produce different and more favorable similarities and differences.

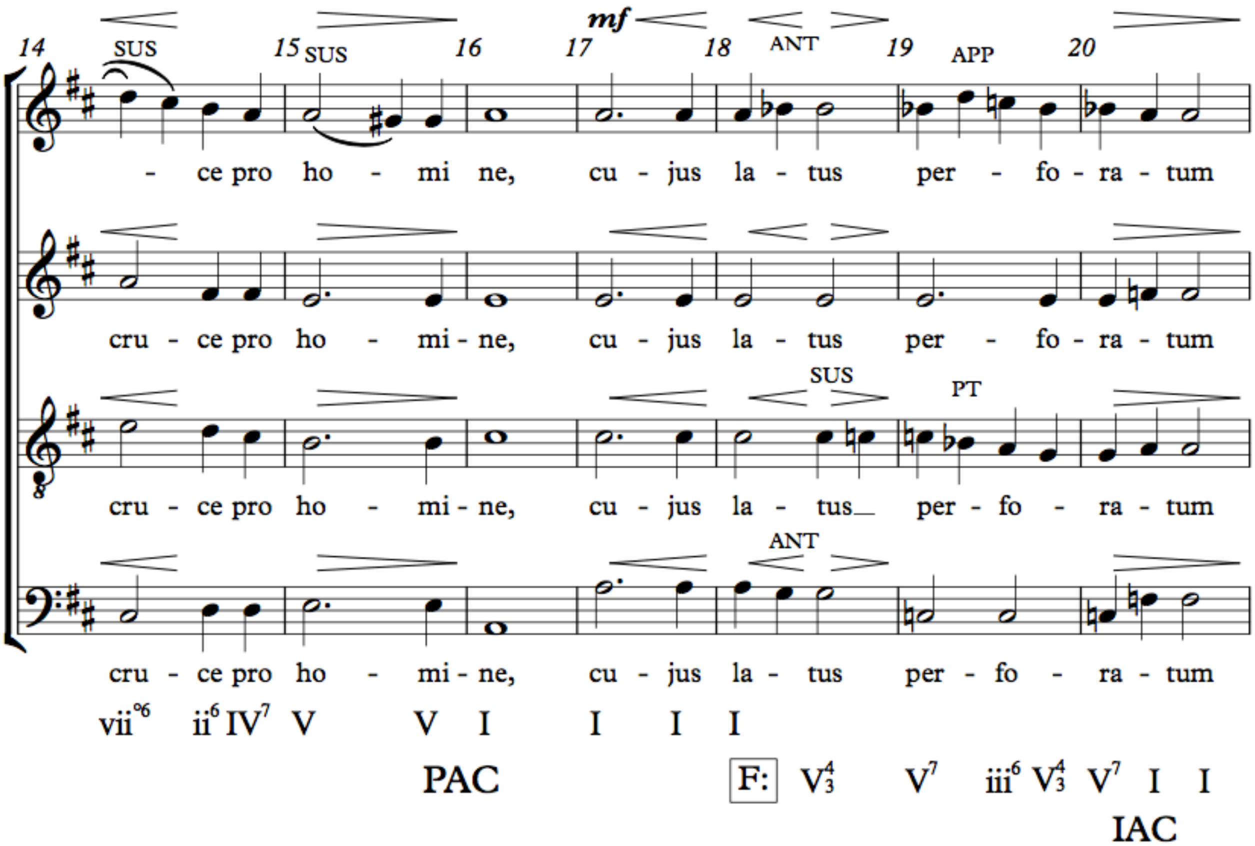
For our topologically transcribed Classical period piece, the piece retained the harmonic rhythm that made the piece interesting, however almost all of the consonances were replaced with dissonances and cadences lost all of their "at home" qualities. It wasn't until the analysis of the Romantic period piece that the reason for this became clear: in Romantic period piece, the forward movement of the piece is provided by non-chord tones, while in the classical period piece all sonorities, consonant and dissonant, drive the music forward. Thus, topological transcription is better for pieces that do not derive their more interesting qualities from non-chord tones. Alternatively, if one were to include the non-chord tones into their topological transcription, perhaps the output might lend itself more towards Romantic style music, since dissonances that do occur in such music are more acceptable in general.

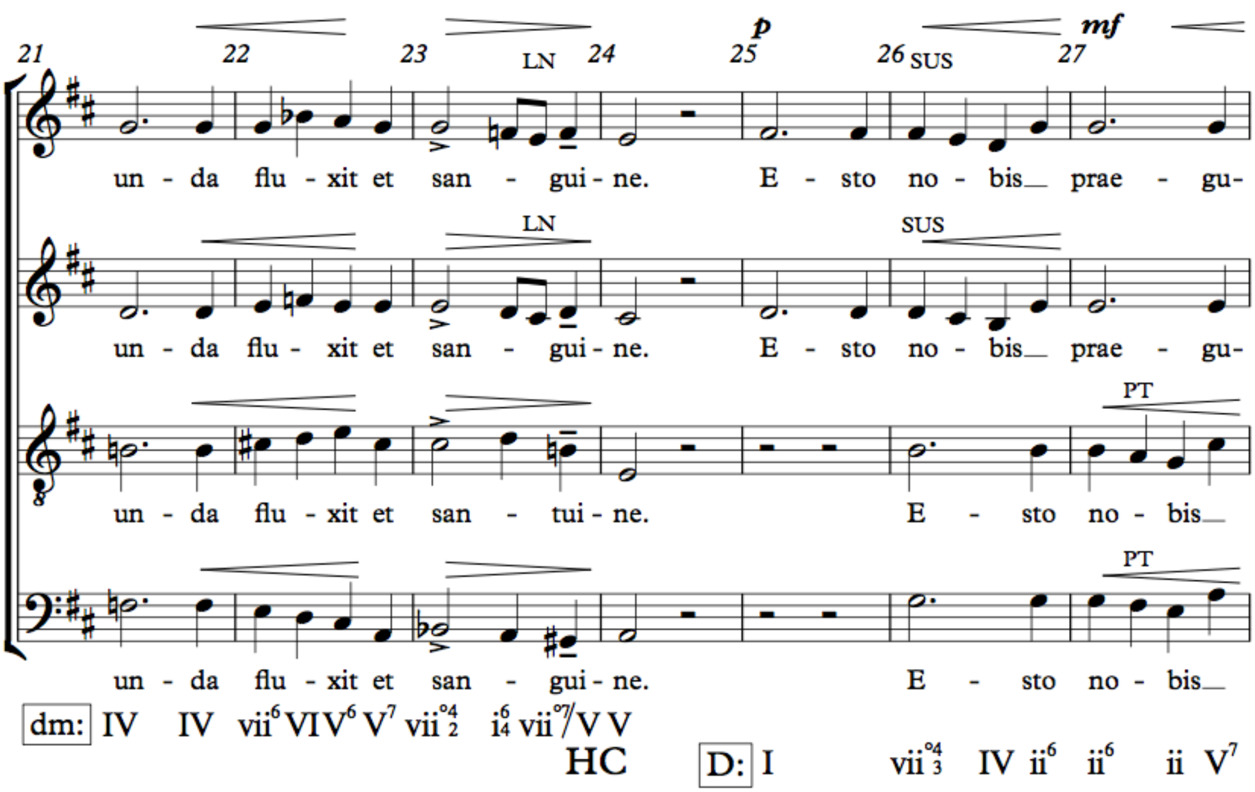
In the Romantic-period transcribed piece, converse to what happened in the Classical transcription, there were many more recognizable chords suggesting tonal centers and valid progressions throughout. Furthermore, some of the identifiable chords were generated originally from extremely unusual chords, which suggests that this inversion inverted the "closeness" of the chord, i.e. reasonable chords became unreasonable and vise versa. Otherwise however, most of the chords remained unrecognizable and, for that reason, the piece was still unsuccessful in its entirety.

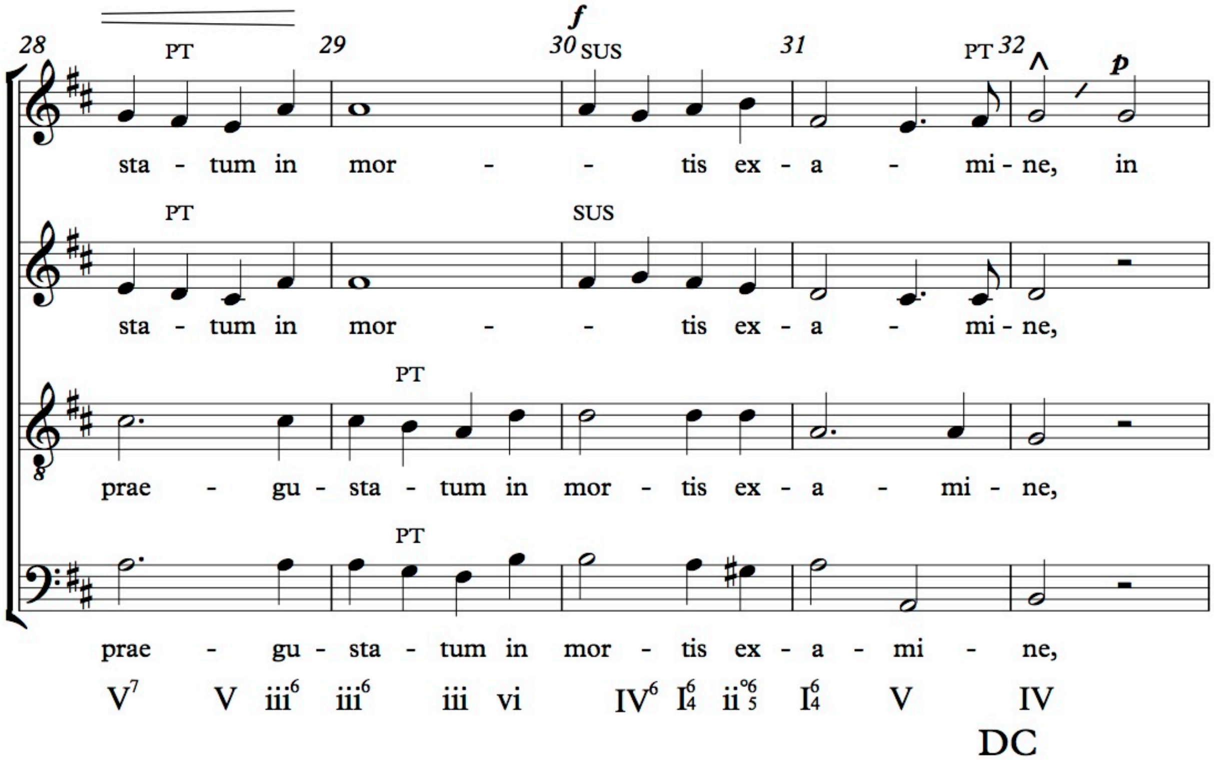
It is clear that there are many drawbacks to composing topologically, the first of which is a separation from music and the text that accompanies it. However, perhaps with more study into the methods and results of topological transcription this effect can be ameliorated. From this paper, some suggestions for further study would be as follows: invert the maps over the $y = x$ axis, include non-chord tones in the topological mapping, and include pieces from the Baroque and Neoclassical periods. These two periods in particular are the preceding and successive periods to those considered in this paper, and thus working outwards there is a potential to study all periods of music topologically. While there isn't much hope for these two transcribed pieces to become music sensations, this paper proves that there is a potential future for topotonal music.

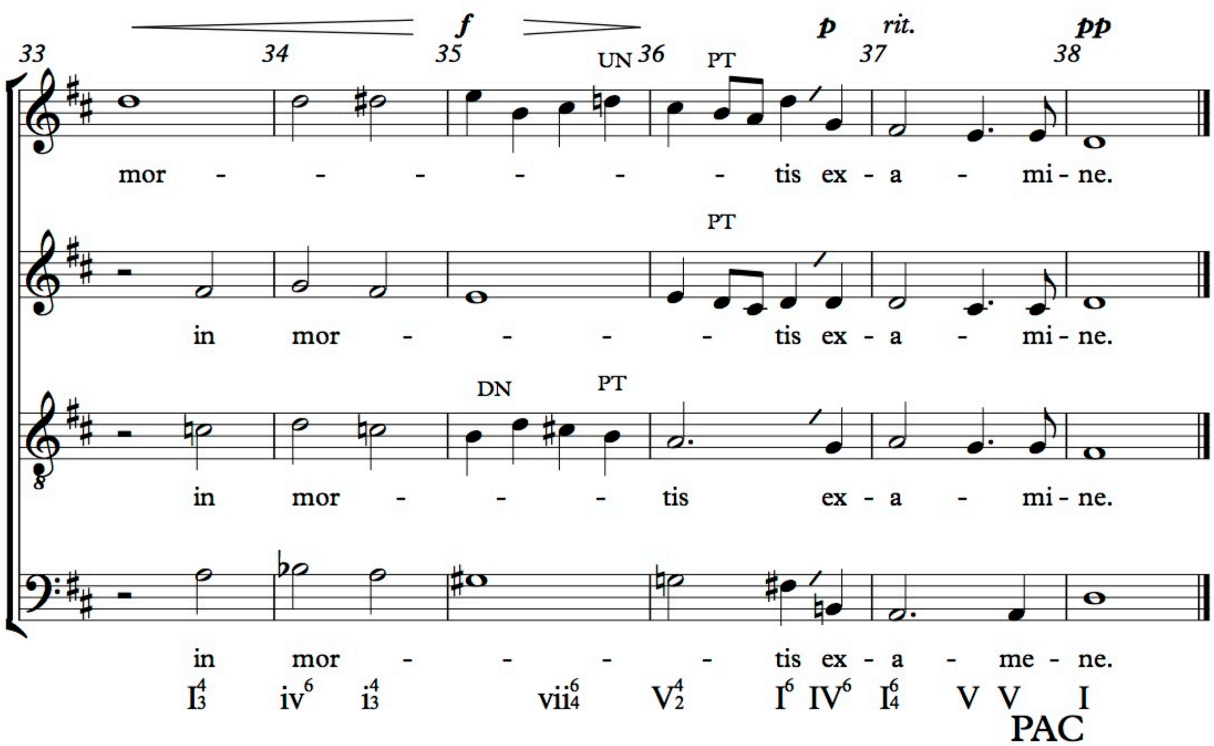
Ave Verum Corpus

W. A. Mozart (1756 - 1791)


mf
14 SUS 15 SUS 16 17 18 ANT 19 APP 20
- ce pro ho - mi ne, cu - jus la - tus per - fo - ra - tum
cru - ce pro ho - mi - ne, cu - jus la - tus per - fo - ra - tum
SUS PT
cru - ce pro ho - mi - ne, cu - jus la - tus per - fo - ra - tum
ANT
cru - ce pro ho - mi - ne, cu - jus la - tus per - fo - ra - tum
vii°6 ii6 IV7 V V I I I I
PAC
F: V4/3 V7 iii6 V4/3 V7 I I
IAC

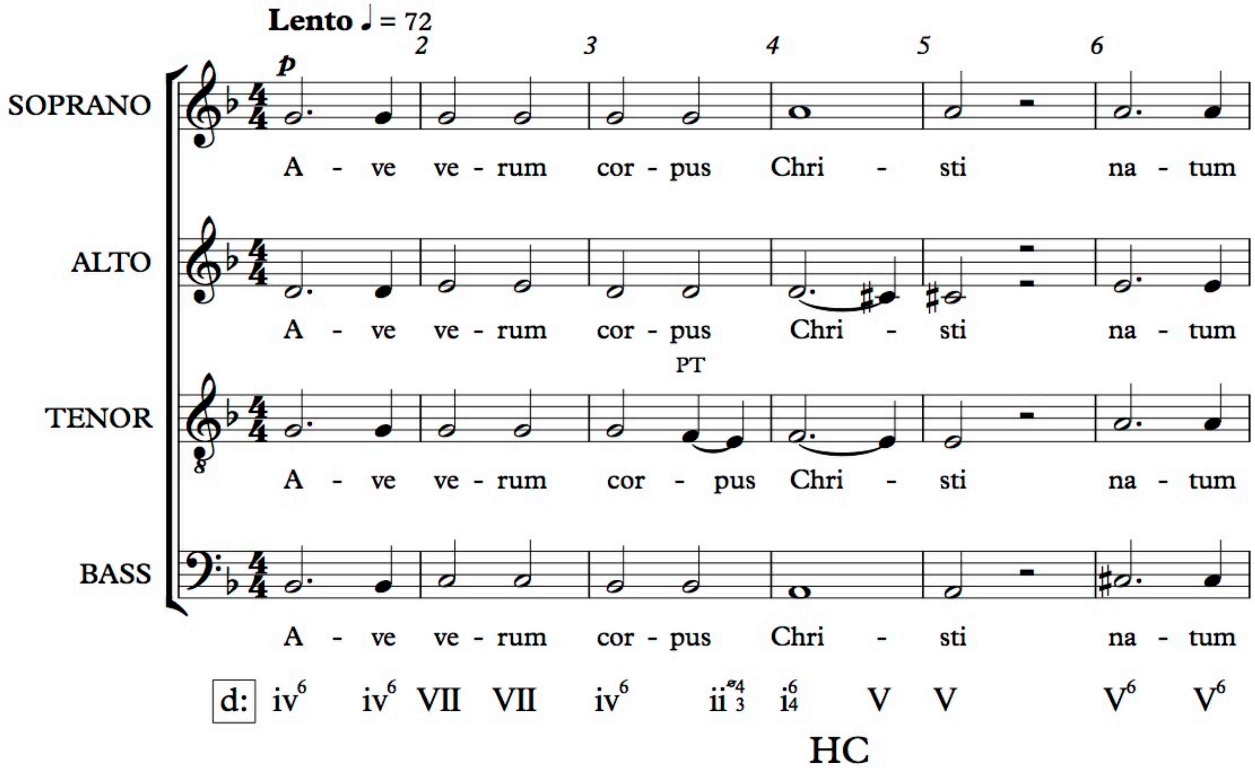

p mf
21 22 23 LN 24 25 26 SUS 27
un - da flu - xit et san - gui - ne. E - sto no - bis prae - gu-
LN SUS
un - da flu - xit et san - gui - ne. E - sto no - bis prae - gu-
PT
un - da flu - xit et san - tui - ne. E - sto no - bis
PT
un - da flu - xit et san - gui - ne. E - sto no - bis
dm: IV IV vii6 VI V6 V7 vii°4/2 i6/4 vii°7/V V
HC
D: I vii°4/3 IV ii6 ii6 ii V7

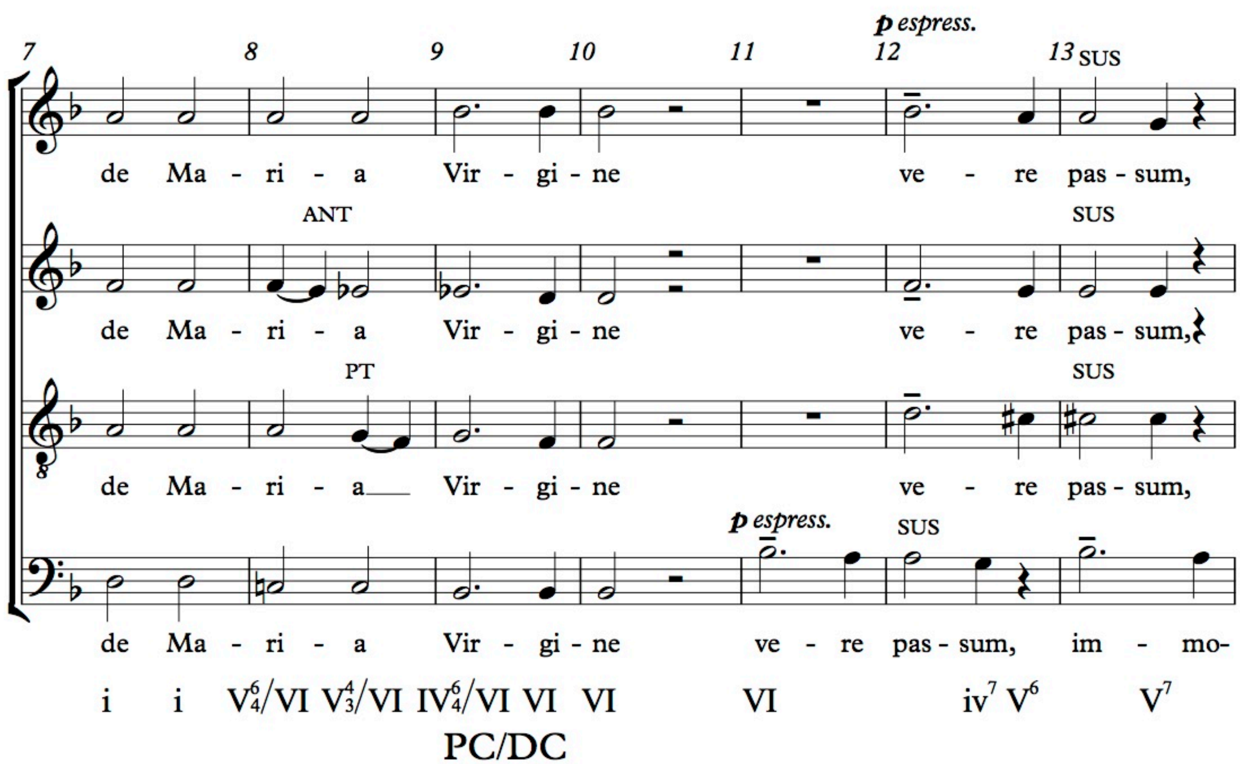

f
28 PT 29 30 SUS 31 PT 32
p
sta - tum in mor - - tis ex - a - mi - ne, in
PT SUS
sta - tum in mor - - tis ex - a - mi - ne,
PT
prae - gu - sta - tum in mor - tis ex - a - mi - ne,
PT
prae - gu - sta - tum in mor - tis ex - a - mi - ne,
V7 V iii6 iii6 iii vi IV6 I64 ii°65 I64 V IV
DC

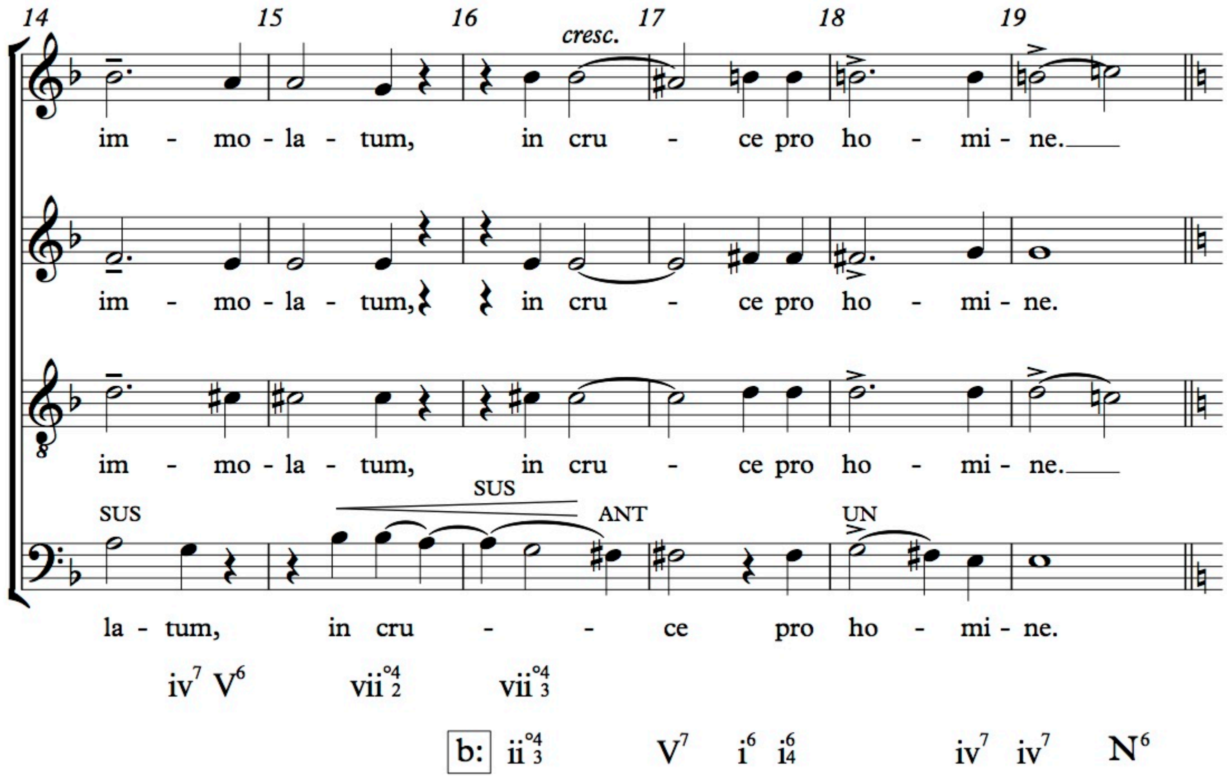

f
p rit. pp
33 34 35 UN 36 PT 37 38
mor - - - - - - tis ex - a - mi - ne.
PT
in mor - - - tis ex - a - mi - ne.
DN PT
in mor - - - tis ex - a - mi - ne.
in mor - - - - tis ex - a - me - ne.
I43 iv6 i43 vii64 V42 I6 IV6 I64 V V I
PAC

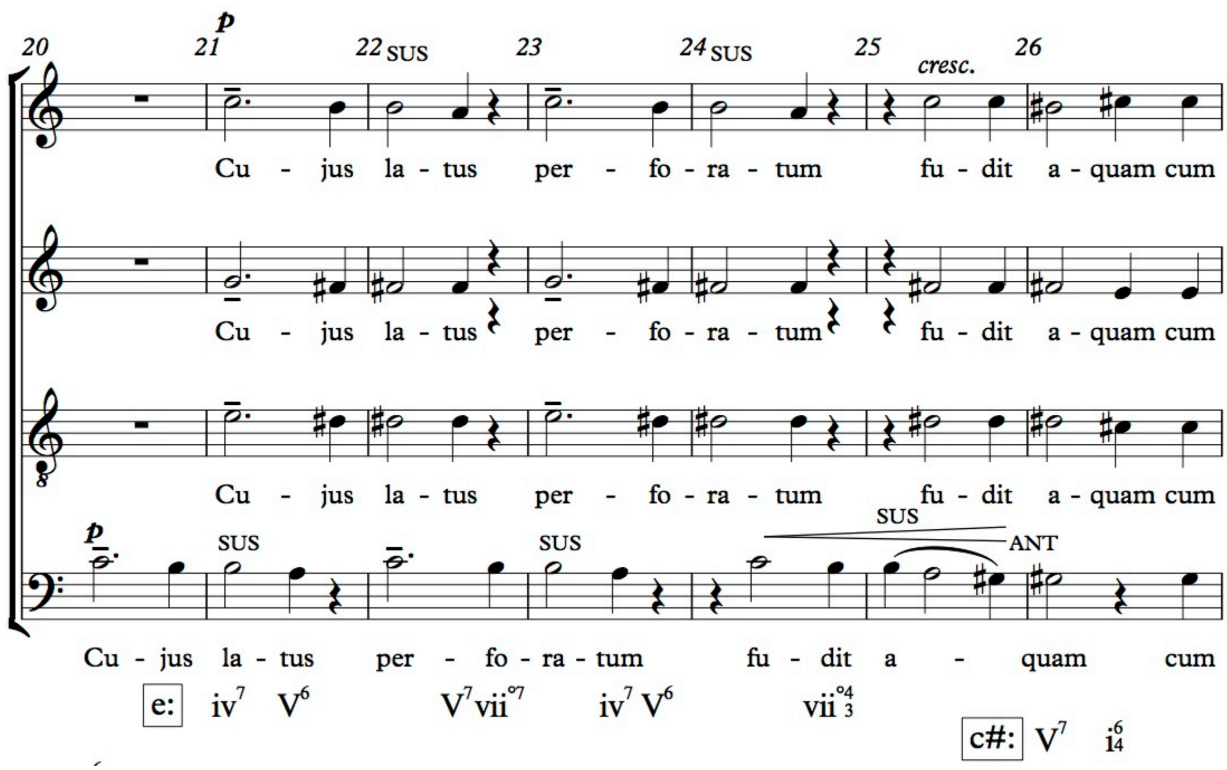
Ave Verum

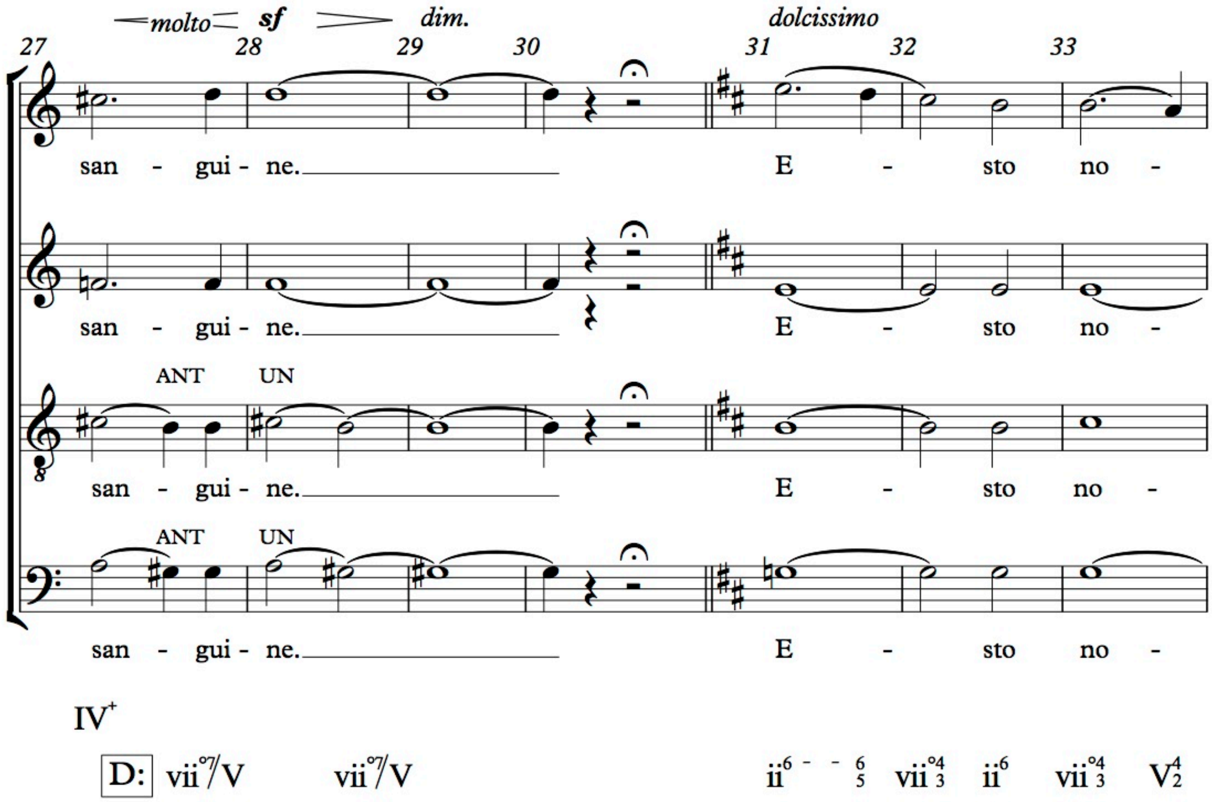
Franz Liszt (1871)

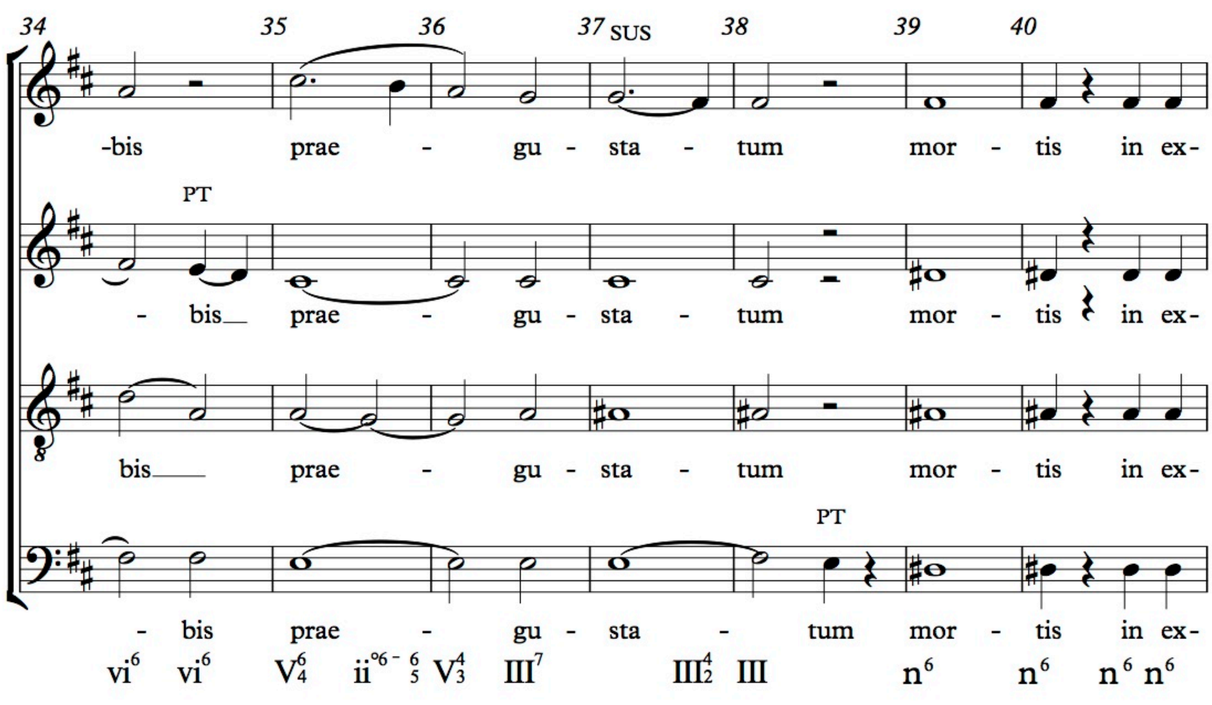


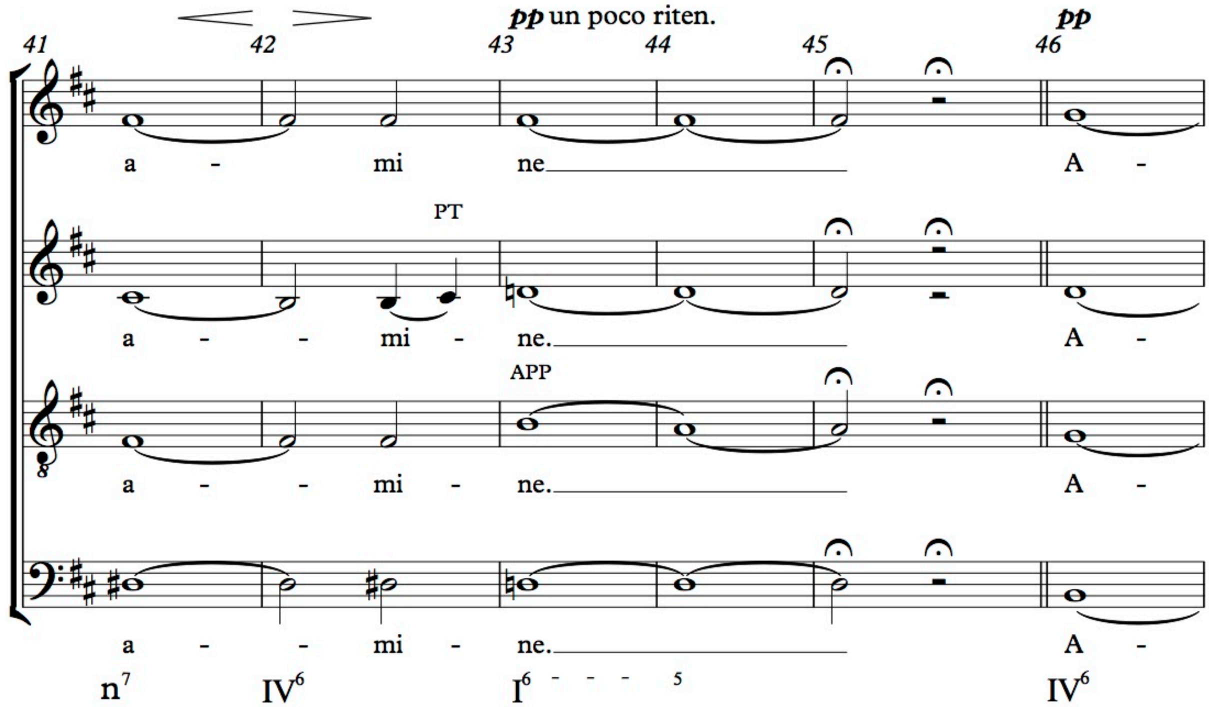


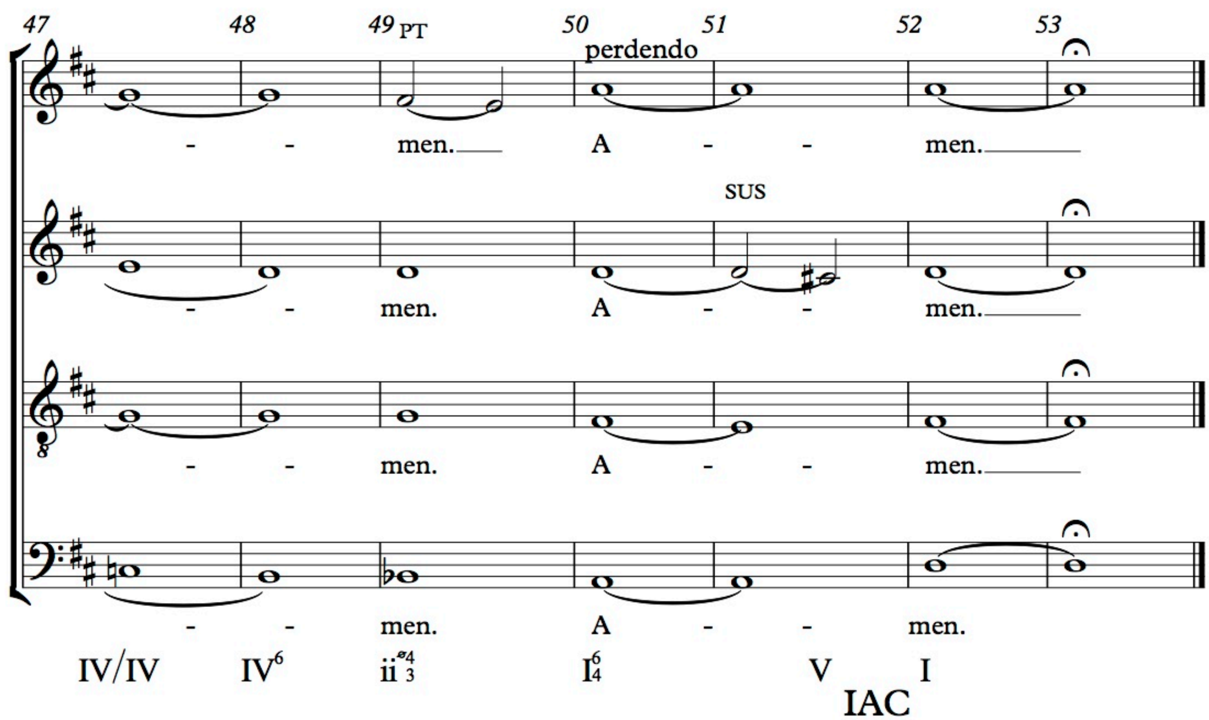

14 15 16 17 18 19
cresc.
im - mo - la - tum, in cru - ce pro ho - mi - ne.
la - tum, in cru - ce pro ho - mi - ne.
SUS
SUS
ANT
UN
iv7 V6 vii°4 2 vii°4 3
b: ii°4 3 V7 i6 i6 4 iv7 iv7 N6


20 21 22 23 24 25 26
p
cresc.
Cu - jus la - tus per - fo - ra - tum fu - dit a - quam cum
Cu - jus la - tus per - fo - ra - tum fu - dit a - quam cum
SUS
SUS
ANT
e: iv7 V6 V7 vii°7 iv7 V6 vii°4 3
c#: V7 i6 4

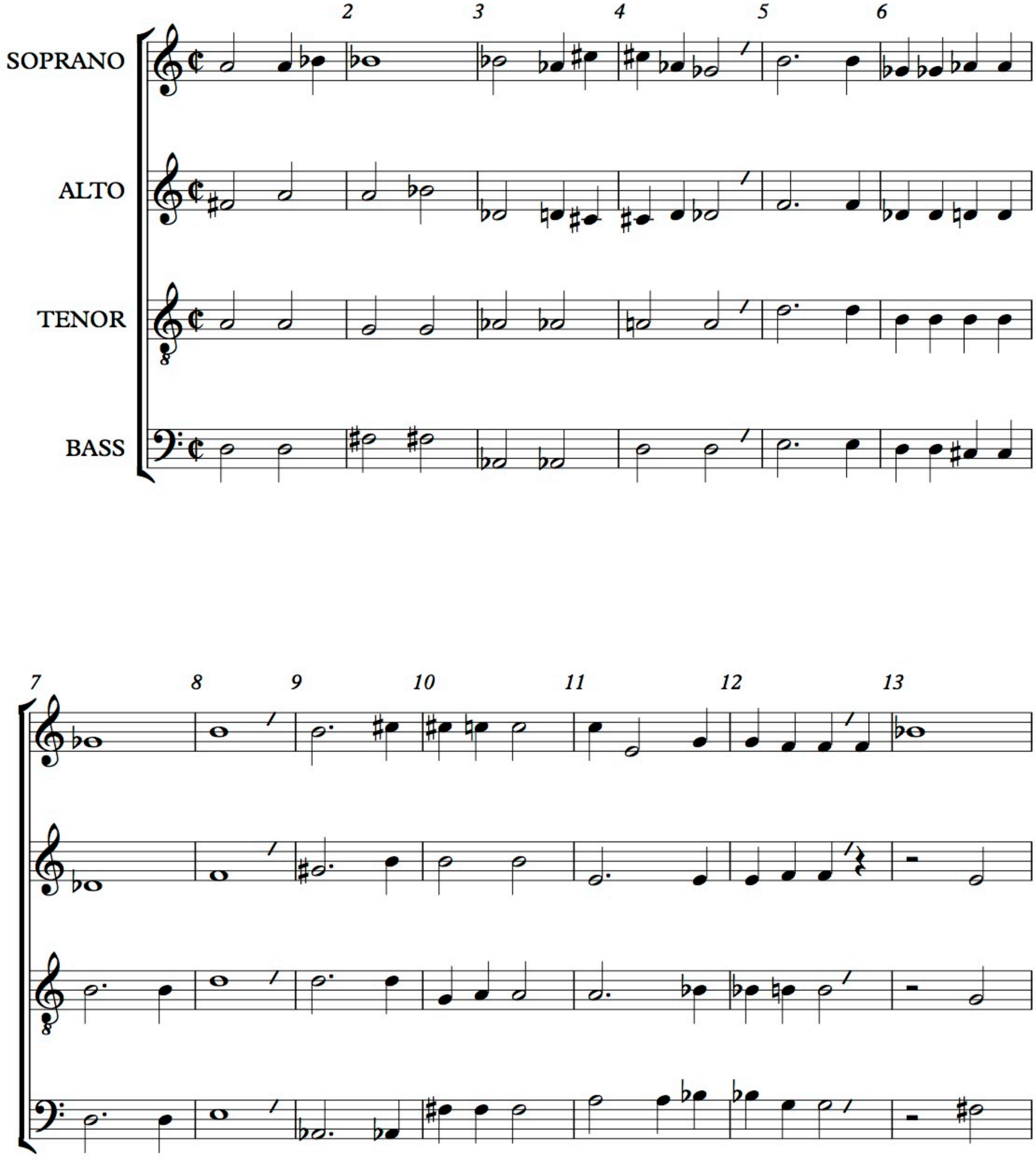

molto
sf
dim.
dolcissimo
san - gui - ne. E - sto no -
ANT
UN
IV+
D: vii°7/V vii°7/V
ii6 ii6/5 vii°4/3 ii6 vii°4/3 V4/2


SUS
-bis prae - gu - sta - tum mor - tis in ex-
PT
bis prae - gu - sta - tum mor - tis in ex-
vi6 vi6 V6/4 ii°6 6/5 V4/3 III7 III4/2 III n6 n6 n6 n6


pp un poco riten.
pp
41
42
43
44
45
46
a - mi ne
A -
PT
a - - mi - ne.
APP
a - - mi - ne.
a - - mi - ne.
n7 IV6 I6 - - - - 5 IV6


47
48
49 PT
50
51
52
53
perdendo
- - men. A - - men.
SUS
- - men. A - - men.
- - men. A - - men.
- - men. A - - men.
IV/IV IV6 ii∅4 3 I6 4 V I
IAC

Topologically Transcribed Classical

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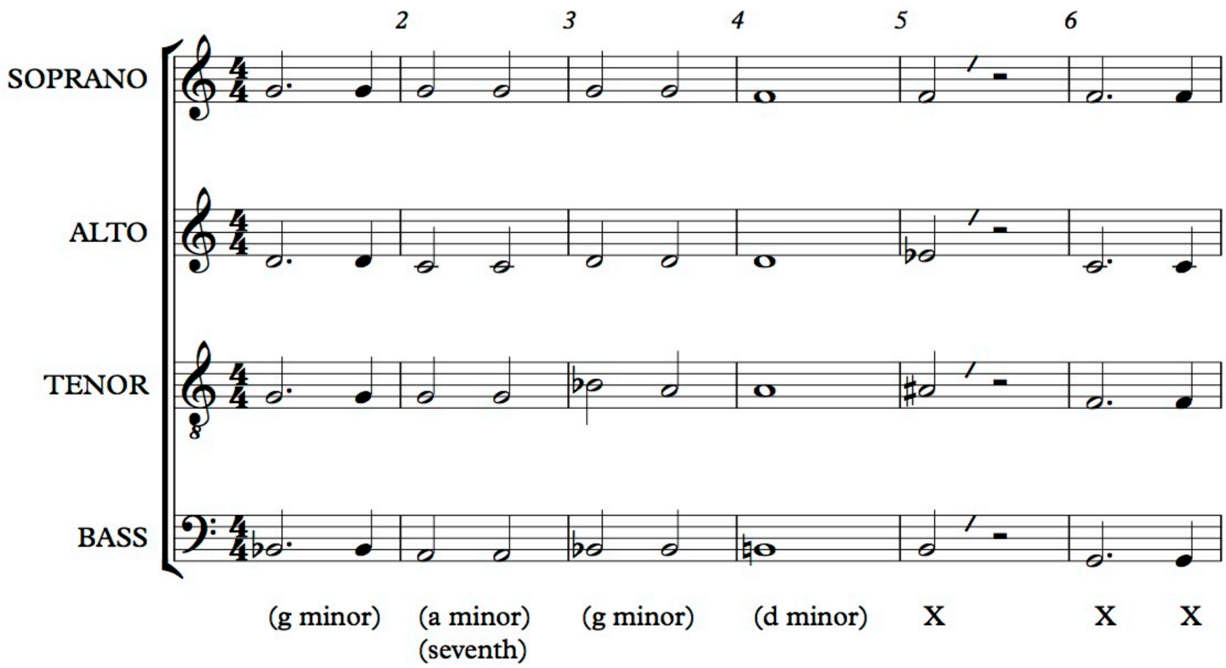
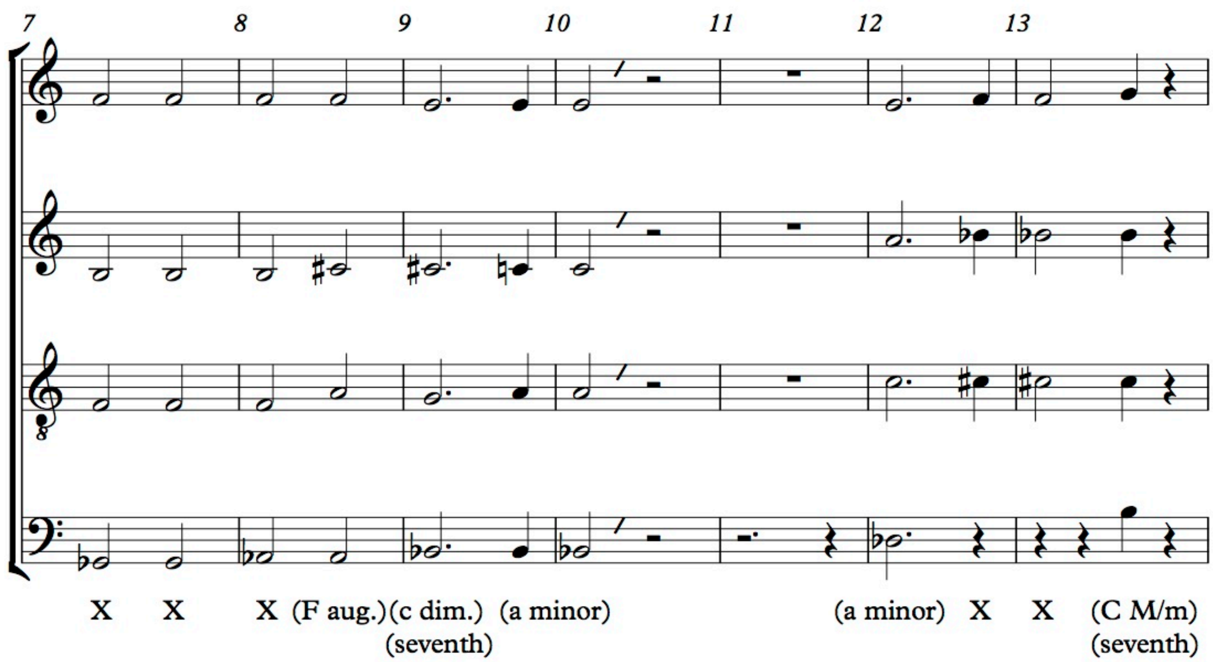
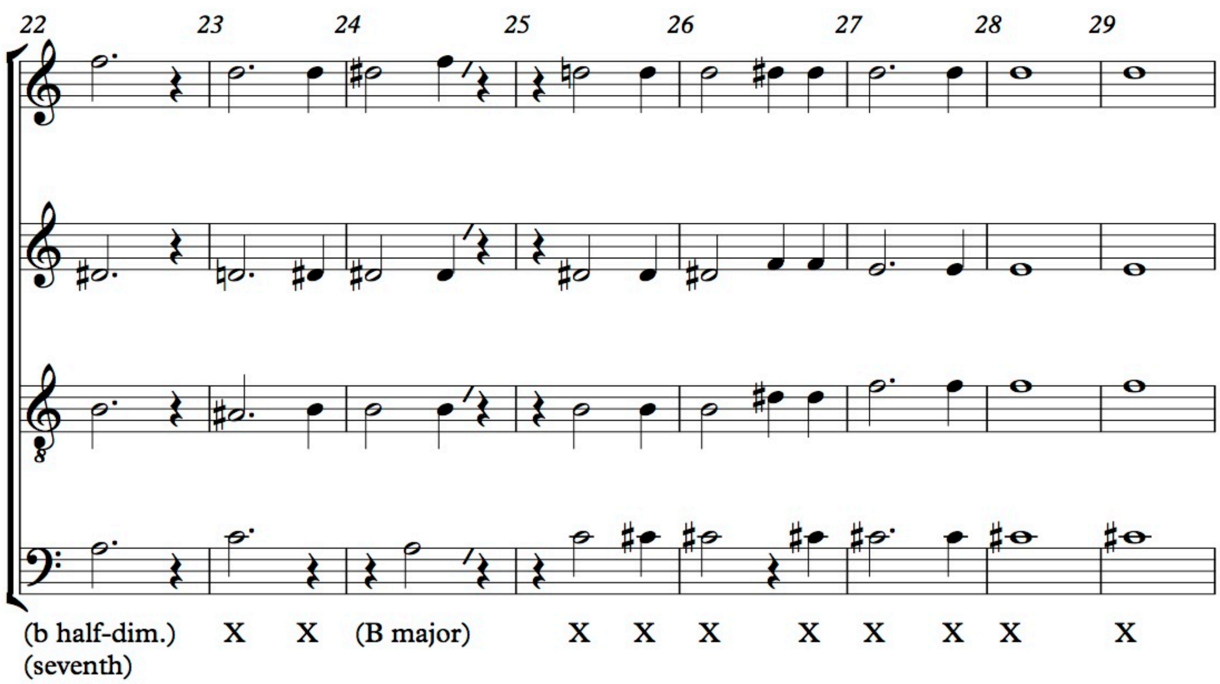
Topologically Transcribed Romantic

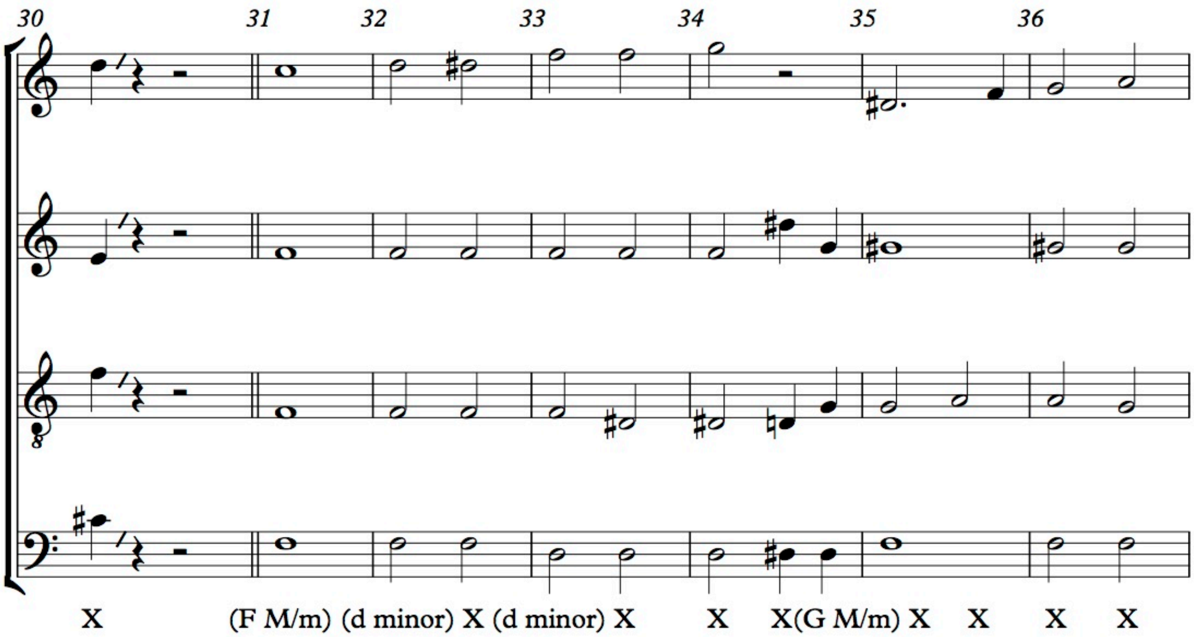
Letizia Campo

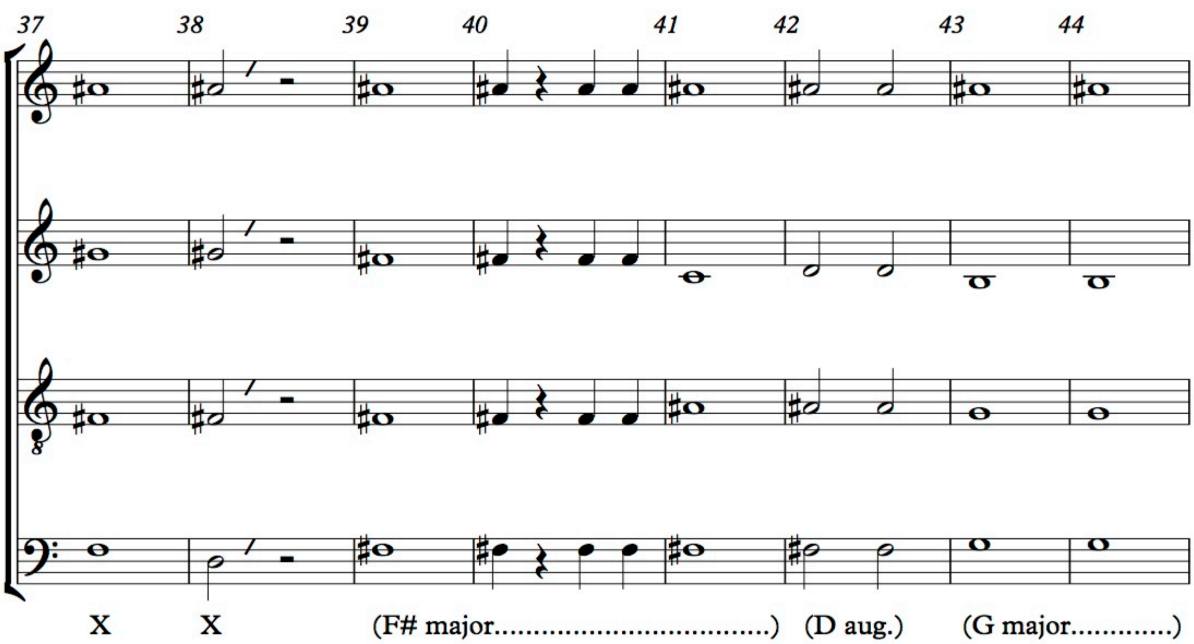


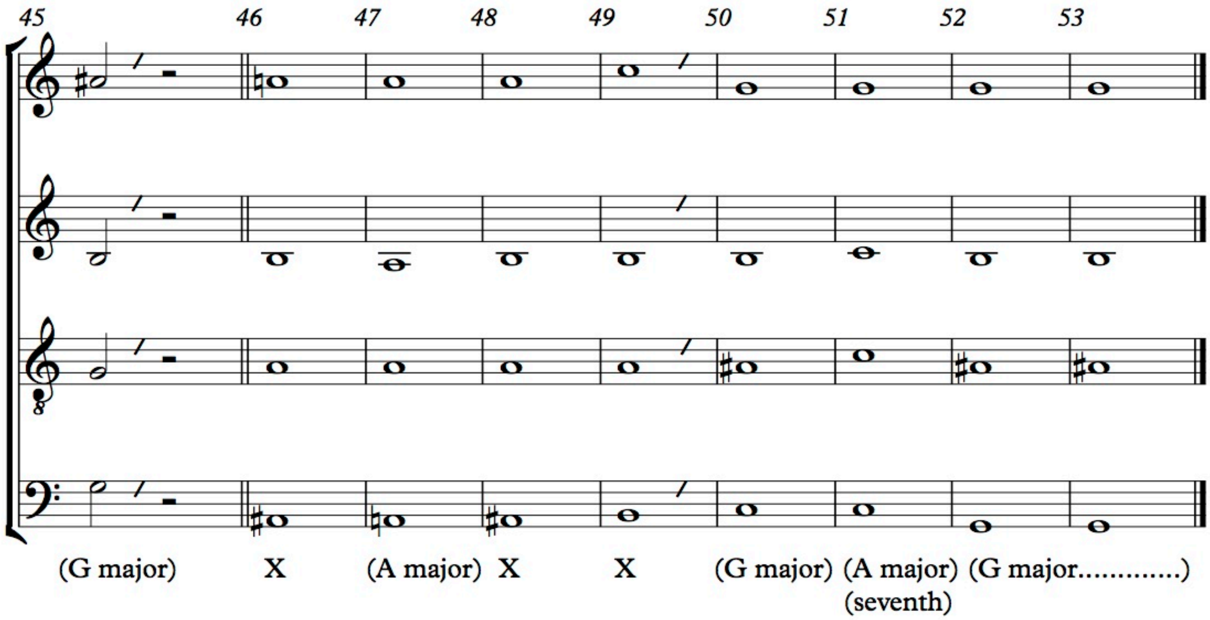



14 15 16 17 18 19 20 21
(a minor) X X X X X X X X X (d minor) X (B major)


22 23 24 25 26 27 28 29
(b half-dim.) X X (B major) X X X X X X X X
(seventh)


30 31 32 33 34 35 36
X (F M/m) (d minor) X (d minor) X X X(G M/m) X X X X


37 38 39 40 41 42 43 44
X X (F# major..................................) (D aug.) (G major.............)


45
46
47
48
49
50
51
52
53
(G major)
X
(A major)
X
X
(G major)
(A major)
(seventh)
(G major.............)

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